Math. 467: Modern Geometry
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First day

Introduce yourself:

Name, major, class

Future plans (career, etc.)

Why are you here? What do you expect to get out of the course?

OBJECTIVES OF THE COURSE

1. To acquaint students with non-Euclidean geometry, a development of the utmost historical and philosophical significance in the development of mathematics.

2. To help develop facility with logical thinking. (This includes, in particular, the distinction between reasoning from abstract axioms containing undefined terms and reasoning from facts about things we already know something about.)

3. To prepare future teachers of high-school geometry to approach Euclidean geometry from a well-informed, modern perspective. (Note: *Not* "to teach how to teach high-school geometry".)

Please read the handout thoroughly; further discussion of class procedures is postponed to Wednesday. Note that the first reading and homework assignment is on the Web, and the written homework is due a week from today. Skip the Preface but read the Introduction and Chapter 1 this week. Be prepared for discussion on Wednesday.

Introduction

Geometry has two aspects:

1. A piece of abstract mathematics, closely related to the vector spaces $\mathbf{R}^2$ and $\mathbf{R}^3$ of linear algebra classes. (The conclusions follow from the definitions.)

2. A theory of physical space, the world we live in. (This theory could be false, or only approximate. The world could be like a sphere, or like a crumpled sheet of paper.)

[Mention homogeneity and global topology.]

Axioms (or *postulates*) have been regarded in two ways:

1. "self-evident truths"

2. arbitrary starting assumptions

Do these correlate with the previous list?

The parallel postulate is the arena where these issues were hashed out. Is it self-evident that (p. 21) "For every line l and every point P that does not lie on l, there exists a unique line m through P that is parallel to l."? [Sketch. Stimulate discussion.]

Definition: (p. 20) Two lines l and m are *parallel* if they do not intersect.

Hilbert axioms — an updating of Euclid for the plane.

Neutral geometry: all the theorems we can prove *without* the parallel postulate.

(This occupies the first half of the course, to midterm exam.)

Hyperbolic geometry: more than one parallel. (This occupies the second half.)

Elliptic geometry: no parallels at all! This is more problematical than hyperbolic (Appendix A).

A *model* is an interpretation of the terms of a theory for which the truth of its axioms can be verified. (See p. 72.) One can find a model for each of the 3 geometries; this shows that the parallel postulate is independent of the others.

Euclidean: $\mathbf{R}^2$.

Elliptic: Crudely, "lines" are great circles on a sphere. They are not infinite, so we see there will be complications.

Hyperbolic: a surface like a saddle or trumpet. (details in late chapters)

Chapter 1

[p. 1: Denounce the epigraph.]

ANCIENT MATHEMATICS

One sometimes hears that the ancient Hebrews believed that $\pi = 3$. [Discuss. Also the Indiana law.]

I Kings 7:23 : And he made a molten sea, ten cubits from the one brim to the other: it was round all about, ... and a line of thirty cubits did compass it round about.

Three new elements (related):

1. proof (not just examples)

2. idealization (Lines are perfectly straight, infinitely thin, and *infinitely long*, although Aristotle taught that the universe must be finite!)

3. math for math's sake (not primarily practical applications)

   p. 4: Where did he get 6 ancient planets?

Euclid's Elements was the basic text for 2000 years. But see epigraph to Ch. 3, p. 103. Diagrams smuggle in assumptions; later we'll introduce *betweenness* concept and axioms to fill the gap.

The goal of this chapter is to give a quick modern cleanup and tour of Euclid's algebra. (I hope it's a review of high school geometry.) Part of the intention is to show that there are still ambiguities that make the rules of the game unclear, therefore motivating our later, slower work based on Hilbert's axioms. So, meanwhile, don't get too hot and bothered.

Primitive terms

- point

- line

- lie on (as relation of point to line)

- between (as relation of point to two other points)

- congruent (replacing ambiguous "equal"): having the same size and shape

Definitions

- segment (between two points)

- circle (with given point as center and given segment as radius)

- rays and angles (see pp. 18–19)

- parallel (nonintersecting) lines   *Note:* This presumes indefinite extendability; lines are not segments.

The 4 noncontroversial postulates

I. Given two distinct points, P and Q, there is a unique line through them (i.e., P and

Q lie on the line).

In Euclid this is stated as a *construction*: "To draw a straight line from any point to any point." In other words, "We shall use the straightedge as a basic tool."

III. Given distinct points O and A, there is a circle with center O and radius OA.

Again, in Euclid this is a construction: "To describe a circle with any center and any distance." In other words, "We shall use the compass as a basic tool."

II. For every segment AB and for every segment CD there exists a unique point E on line $\overleftrightarrow{AB}$ such that B is between A and E and segment CD is congruent to segment BE.

Euclid says, "To extend a finite straight line continuously in a straight line." In modern times it is often paraphrased as, "A straight line can be extended indefinitely in either direction," and the parallel postulate does not make much sense without this understanding. Presumably Greenberg has stated the postulate in the precise form that Euclid needs to use it later.

IV. All right angles are congruent.

Recall that an angle is defined to be a right angle if it has a supplementary angle to which it is congruent. We don't have "degrees" yet.

Euclid's version is the same, except for "equal to one another" instead of "congruent". But here is where the historical controversies began: Proclus argued that IV can be proved from the other postulates, and we'll see (p. 143) that that is true in the Hilbert reformulation.

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THE PARALLEL POSTULATE

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P. For every line l and every P not on l, there is a unique line m through P parallel to l.

Euclid's fifth postulate was superficially very different:

E V. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Later geometers (notably Proclus and Playfair) settled on P as the best formulation.

[Proceed to the class exercises and discussions from Chap 1.]