Chapter 2

Logic this week, incidence geometry next.

Axiomatic systems

(See "requirements" on pp. 10–11 and pp. 53–58 in Ch. 2.)

The old, naive view of mathematical rigor (seemingly implicit in Euclid):

1. Define all terms.
2. Deduce all statements from self-evident truths.

We have seen that these both lead to problems.

Deduction: We need *two* starting elements:

1. Axioms: "certain statements accepted without further justification".

2. Rules of logical deduction: ways of passing from old statements to new ones.

From these, new statements (theorems) follow. Both are needed to avoid infinite regress. Without the logic rules, the process would be stalled. Without the axioms, there would be no place to start.

Rigorous axiomatization is usually applied to a *mature* subject, where there is already some agreement on the basic facts. It is not so much a powerful way of discovering truth as a way of organizing, systematizing, and applying quality control.

In the modern view, the axioms are usually not "self-evident".

1. Sometimes the axioms are *arbitrary* assumptions, whose consequences we want to deduce, without necessarily caring about whether the statements are "true" of anything in the "real world". (Example: very abstract topics in pure algebra; the axioms are part of the *definition* of a type of mathematical system.)
2. Sometimes, when we *are* seeking a theory *of* something, the verification is of the structure as a whole. In the choice of axioms, power and simplicity are valued over obviousness. (Examples: set theory; theoretical physics.)

Terms: Euclid notoriously started with these definitions:

1. A *point* is that which has no part. (??)
2. A *line* is breadthless length.
3. The extremities of a line are points. (What does this define?? Heath does not emphasize any word.)

4. A *straight line* is a line which lies evenly with the points on itself. (??)

   . . .

12. An *acute angle* is an angle less than a right angle. (That's more like it!)

    Again we must avoid an infinite regress by allowing another starting element:

0. Primitive (undefinable) terms.

Just as a logical deduction produces new theorems from old, a definition produces new terms from old. One can look at that in two ways:

Abbreviation: We just want to make statements shorter.

Explication: By careful thought, we determine what we *really mean* by terms we may have been using informally already.

Similarly, a theorem has two functions:

Proof: We believe the statement if we believe the assumptions.

Explanation: We already thought the statement was true, but now we feel we know *why*, because it fits into the system. (This is the "retail" version of "verifying the structure as a whole".)

Quantifiers

$\forall x$ means "For all x":

$$\forall x\,[x < x + x^2 + 1] \qquad \text{(real numbers understood)}$$

is a true statement. $\exists x$ means "There exists an x such that," as in

$$\exists x\,[x \text{ is rotten in Denmark}].$$

$\forall x$ and $\exists x$ are called *universal* and *existential quantifiers*, respectively.

Expressions such as

$$x < x + x^2 + 1 \qquad \text{and} \qquad x \text{ is rotten in Denmark}$$

(which contain a variable and would be sentences if the variable were replaced by a meaningful name or noun) are called *open sentences*. They are just like *functions* or *formulas* in algebra and calculus, except that the *value* of such a function, when something particular is plugged in for x, is not a number, but rather a *truth value* — either True or False (or either "Yes" or "No").

A quantifier, $\forall x$ or $\exists x$, closes off an open sentence and turns it into a genuine sentence, which is either true or false (although we may not know which). Quantifiers are very much like the definite integral and limit notations in calculus, which turn formulas into numbers:

$$\int_0^1 x^2\,dx \qquad \text{and} \qquad \lim_{x\to 2} x^2$$

are particular numbers, even though the expressions representing them involve a variable, x. In calculus such a variable is often called a "dummy variable"; in logic it's traditionally called a "bound variable" (because it's tied to its quantifier).

The quantified variable stands for objects in some "universe of discourse", which may be either stated explicitly —

$$\forall x\,[\text{if } x \text{ is a real number, then } x < x + x^2 + 1]$$

— or understood from context. The universe of discourse is just like the "domain" of a numerical function.

Leading universal quantifiers are often omitted when we state "identities" in mathematics, such as

$$x + y = y + x \qquad \text{and} \qquad (n+1)! = (n+1)n!\,.$$

[What universe of discourse is understood from context in each of these cases?]

If two or more quantifiers of the same type are adjacent, their order doesn't matter:

$$\forall x \forall y\,[x + y < y + x + 1] \qquad \text{and} \qquad \forall y \forall x\,[x + y < y + x + 1]$$

say exactly the same thing.

However, the order of quantifiers of different type is extremely important. (This is the place where this discussion gets beyond the obvious into something both important and subtle.) Consider, for example, the old saying

Behind every successful man there is a woman.

The structure of this proposition (whether or not you believe it to be true or false) is

$$\forall x \exists y\,[\text{if } x \text{ is a man and } x \text{ is successful, then } y \text{ is a woman and } y \text{ is behind } x].$$

But

$$\exists y \forall x\,[\text{if } x \text{ is a man and } x \text{ is successful, then } y \text{ is a woman and } y \text{ is behind } x]$$

says something completely different: There is *one particular woman* who stands behind every successful man in the world! (There may be more than one, but each one of them deals with all the men.)

It is notorious that this point is important for the ϵ and δ in the definition of a limit,

> The function f is continuous if, for every x in the domain of f, for every number $\epsilon > 0$ there is a number $\delta > 0$ such that for any number y in the domain, if $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. (1)

which has this structure:

$$\forall x \forall \epsilon \exists \delta \forall y \, [\text{if } |y - x| < \delta, \text{ then } |f(y) - f(x)| < \epsilon] \tag{1'}$$

(with qualifications such as "number" and "in the domain" suppressed). For example, look at this slight variation of (1):

> For every x in the domain of f, there is a number $\delta > 0$ such that for every number $\epsilon > 0$ and every number y in the domain, if $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. (2)

Its structure is

$$\forall x \exists \delta \forall \epsilon \forall y \, [\text{if } |y - x| < \delta, \text{ then } |f(y) - f(x)| < \epsilon], \tag{2'}$$

and it is false unless f is a constant function! For a nontrivial f, one has to know ϵ before one can choose the right δ. There is not (usually) one δ that works for every ϵ. Similarly, if we move the universal quantifier $\forall x$ in (1) after the existential quantifier $\exists \delta$, we get a different condition:

> For every number $\epsilon > 0$ there is a number $\delta > 0$ such that for every x and y in the domain of f, if $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. (3)

$$\forall \epsilon \exists \delta \forall x \forall y \, [\text{if } |y - x| < \delta, \text{ then } |f(y) - f(x)| < \epsilon]. \tag{3'}$$

Although this condition is simpler to state in English than the one in the definition (1), it is *harder* for the function f to satisfy, because it requires that the *same* δ work for all x. For example, the function $f(x) = 1/x$ is continuous on the domain $(0, \infty)$, but it does not satisfy (3) because its graph becomes increasingly steep as x approaches 0. A function that *does* satisfy (3) is called *uniformly continuous*. Uniform continuity is an important condition in more advanced mathematics, but we are interested in it today only as an example of the need to keep track of the order of quantifiers. (By the way, Augustin-Louis Cauchy, who did as much as anyone to invent the concept of a limit, seems to have been confused on this point for at least 26 years, so students shouldn't expect to find it easy on the first exposure.)

If two quantifiers of one type are separated by one (or more) of the other type, then they cannot be reversed:

$$\exists x \forall y \exists z \quad \text{is not equivalent to} \quad \exists z \forall y \exists x;$$

"There is a country where behind every man there is a woman" is not equivalent to "There is a woman such that for every man there a country where she stands behind him."

Propositional calculus (the logical connectives)

Letters $p, q, \ldots$ are used as variables standing for statements or for open sentences (sentences containing variables). In the latter case, the variable is sometimes made explicit by writing something like $p(x)$. (For example, we can let p, or $p(n)$, stand for "n is a perfect square" — i.e.,

$$\exists m \, [n = m^2]$$

— and q for "n is an even number." To discuss p in the context of quantifiers, we would write $\exists n \, p(n)$, etc., or even $\exists n \exists m \, r(n, m)$ to display the entire structure.) The second major part of logical notation expresses how simple sentences are combined into compound ones. The easiest of these to understand simply translate the English words "and", "or", and "not".

AND: $p \wedge q$ (n is a square and also is even.)

OR: $p \vee q$ (Either n is a square, or it is even (possibly both).)

NOT: $\neg p$ (n is not a perfect square.)

Note that $\neg\neg p$ simplifies to p. Also, it is easy to see that $\wedge$ and $\vee$ are commutative and associative operations, so we can write things like $p \wedge q \wedge r$ instead of $p \wedge (q \wedge r)$ and $(p \wedge r) \wedge q$. Furthermore, each of them is distributive over the other:

$$p \wedge (q \vee r) \quad \text{is equivalent to} \quad (p \wedge q) \vee (p \wedge r), \tag{4a}$$

$$p \vee (q \wedge r) \quad \text{is equivalent to} \quad (p \vee q) \wedge (p \vee r). \tag{4b}$$

(We shall prove one of these with truth tables in a moment.)

Truth tables

It is convenient and standard to let 1 represent "True" or "Yes" and 0 represent "False" or "No".

Each connective can be precisely defined by telling what its truth value is for each possible truth value of its parts. This information can be presented in *truth tables* similar to addition and multiplication tables in arithmetic:

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

From these the truth tables of more complicated sentences can be deduced. For example, let's establish the distributive law (4a). We list all 8 possible cases:

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	0	0	0	0	0	0
1	0	1	1	1	0	1	1
1	1	0	1	1	1	0	1
1	1	1	1	1	1	1	1

From the fifth and eighth columns of this table, we see that the two sides of (4a) are true in exactly the same cases (the last 3); therefore, replacing one side by the other is a universally valid principle of logical reasoning.

De Morgan's laws

Two other logical identities that can be verified by truth tables are *De Morgan's laws*,

$$\neg(p \wedge q) \quad \text{is equivalent to} \quad \neg p \vee \neg q, \tag{5a}$$

$$\neg(p \vee q) \quad \text{is equivalent to} \quad \neg p \wedge \neg q. \tag{5b}$$

Let's look now at what happens when we try to prove an identity that turns out to be incorrect. Suppose we conjectured that

$$\neg(p \wedge q) \quad \text{is equivalent to} \quad \neg p \wedge \neg q.$$

We work out the corresponding truth table:

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
0	0	0	1	1	1	1
0	1	0	1	1	0	0
1	0	0	1	0	1	0
1	1	1	0	0	0	0

The fourth and seventh columns do not match up, so the conjecture was wrong.

Closely related to De Morgan's laws are these laws for quantifiers:

$$\neg\forall x\,[\ldots] \quad \text{is equivalent to} \quad \exists x \neg[\ldots], \tag{6a}$$

$$\neg\exists x\,[\ldots] \quad \text{is equivalent to} \quad \forall x \neg[\ldots]. \tag{6b}$$

Note that for a *finite* universe, quantifiers are unnecessary — they can be rewritten in terms of connectives! If the universe has only 3 elements, say a, b, c, then

$$\forall x\, p(x) \quad \text{is equivalent to} \quad p(a) \wedge p(b) \wedge p(c),$$

$$\exists x\, p(x) \quad \text{is equivalent to} \quad p(a) \vee p(b) \vee p(c).$$

Then the action (6) of $\neg$ on quantifiers follows from DeMorgan's laws (5).

Both of these sets of laws are especially nice when there are negations on both sides:

$$\neg(\neg p \wedge \neg q) \quad \text{simplifies to} \quad p \vee q,$$

$$\neg \forall x\, \neg p(x) \quad \text{simplifies to} \quad \exists x\, p(x),$$

etc.

Connectives expressing logical equivalence and implication

We consider two more extremely important logical connectives:

IFF:

p	q	$p \longleftrightarrow q$
0	0	1
0	1	0
1	0	0
1	1	1

ONLY IF:

p	q	$p \rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

("p ONLY IF q" is the same as "q IF p".)

Remarks:

1. $p \longleftrightarrow q$ says that p and q have the *same truth value* (either both true or both false). Therefore, if we know one, we can conclude the other. The distributive law (4a) can be expressed totally in symbolic notation as

$$p \wedge (q \vee r) \longleftrightarrow (p \wedge q) \vee (p \wedge r).$$

2. $p \rightarrow q$ is intended to symbolize that *if we know that p is true, then we can conclude that q is true*. Note that the bottom half of its truth table guarantees this. With the full set of logical notation, we can now write the structure of the definition of continuity entirely in symbols:

$$[f \text{ is continuous}] \longleftrightarrow \forall x \forall \epsilon \exists \delta \forall y\, [p(x,y,\delta) \rightarrow q(x,y,\epsilon)]. \qquad (1'')$$

3. The most common English rendering of $p \rightarrow q$ is

$$\text{If } p, \text{ then } q, \qquad (7)$$

One also says "p implies q." It is sometimes said that the English counterpart of $\rightarrow$ is IF, but this is not quite right, because the "if" in (7) comes *before* the p. Therefore,

$$p \text{ IF } q$$

corresponds better to $q \Rightarrow p$ than to $p \Rightarrow q$. However, if we say

$$p \text{ ONLY IF } q,$$

then we do get something that means $p \rightarrow q$; it says that if q is false, then p is false, which is the contrapositive (see 8 below) of $p \rightarrow q$.

4. $p \longleftrightarrow q$ is equivalent to $(p \rightarrow q) \wedge (q \rightarrow p)$. Thus

$$p \text{ IF AND ONLY IF } q$$

is an English way of saying $p \longleftrightarrow q$. In fact, it is the standard way of expressing equivalent conditions (usually $\forall x\,[p(x) \longleftrightarrow q(x)]$ statements) in mathematical English. Often it is "blackboard abbreviated" to "IFF".

5. $p \rightarrow q$ is equivalent to $\neg p \vee q$. (By the way, like the minus sign in algebra, $\neg$ binds tightly to the propositional symbol it applies to ("has highest precedence"); for example, $\neg p \vee q$ means $(\neg p) \vee q$, not $\neg(p \vee q)$.)

6. The connective $\rightarrow$ is slightly subtle conceptually. Right now you may be wanting to ask:

   A. What is the justification for the top half of the table? Does it make sense to say, "If $2+2=5$, then $\sin x$ is a continuous function," or "If $2+2=5$, then $\sin x$ is not a continuous function."?

   B. Does it make sense to say, "If China is in Asia, then $\sin x$ is a continuous function," when the two statements obviously have no connection with each other?

The answer to question A is that we want $p \rightarrow q$ to be meaningful and useful when p and q are *open sentences* — in particular, when they are inside *quantified* sentences, such as

$$\forall x \left[\text{If } |x| < \frac{\pi}{4}, \text{ then } |\sin x| < \frac{1}{\sqrt{2}}\,.\right]$$

This is a true and useful theorem. It does what we want of a theorem: Whenever a number is less than $\frac{\pi}{4}$ in magnitude, it enables us to conclude (correctly) that its sine is less than $\frac{1}{\sqrt{2}}$. However, there are other numbers, such as $x = \frac{\pi}{2}$, for which both the hypothesis and the conclusion are false, and there are still other numbers, such as $x = \pi$, for which the hypothesis is false but the conclusion is true. We must demand that these cases be *consistent with the theorem*, and the connective $\rightarrow$ is defined to make this so. If we changed the top two lines of the truth table, or left them undefined, then the theorem would become false, or indeterminate, for some of these cases. This would make the formulation of mathematical statements very cumbersome.

The resolution of point B is similar. Propositional calculus is concerned only with the *truth values* of sentences, not with what they *mean*. There are only 2 truth values, Yes and No. In this sense all true sentences are the same, and all false ones are the same, just as all numbers 9 are the same, regardless of what things you counted to get the number 9. Therefore, to say

$$\text{If China is in Asia, then } \sin x \text{ is a continuous function.} \tag{8}$$

is no more strange than to say

$$\begin{gathered}\text{The number of planets in the solar system is less than the number}\\ \text{of states in the Union.}\end{gathered} \tag{9}$$

There is no scientific law that makes (9) true; it is simply a fact. The same goes for (8).

7. If $p \Rightarrow q$, or $\forall x\,[p(x) \rightarrow q(x)]$, then one says

$$p \text{ is a } \textit{sufficient condition} \text{ for } q$$

and

$$q \text{ is a } \textit{necessary condition} \text{ for } p$$

(that is, if q is false, then p can't be true). Therefore, if p is both necessary and sufficient for q, then $p \iff q$ or $\forall x\,[p(x) \longleftrightarrow q(x)]$.

8. Associated with $p \rightarrow q$ are 3 closely related propositions:

$$\begin{array}{cc} \text{contrapositive:} & \neg q \Rightarrow \neg p \\ \text{converse:} & q \Rightarrow p \\ \text{inverse:} & \neg p \Rightarrow \neg q \end{array}$$

The original statement and its contrapositive are logically equivalent. The converse and the inverse are equivalent, because the inverse is the contrapositive of the converse. But the original and the converse are *not* logically equivalent (although they may both be true under certain circumstances).

Example of 7 and 8: Another bugaboo of the calculus student is *infinite series*, $\sum_{n=0}^{\infty} a_n$. Let p stand for

$$a_n \to 0 \quad \text{as } n \to \infty,$$

and q stand for

$$\text{The series } \sum_{n=0}^{\infty} a_n \text{ converges.}$$

Then q implies p (Stewart, Sec. 10.2, Theorem 6), but p does *not* imply q (because the harmonic series, for instance, is a counterexample: Stewart Sec. 10.2, Example 7). Thus

(A) Tending of the terms to 0 is a *necessary* condition for convergence, but not a *sufficient* condition. (In contrast, most of the series convergence theorems state *sufficient* conditions (e.g., the alternating series test (Stewart, Sec. 10.4) and the ratio test (Sec. 10.3).)

(B) $p \rightarrow q$ is false for the harmonic series, but its converse, $q \rightarrow p$, is true. (Of course, $p \rightarrow q$ is true of all series that happen to be convergent.) If we attach a universal quantifier over all series, $a \equiv \{a_n\}$, then the result

$$\forall a\,[p(a) \rightarrow q(a)]$$

is false, but

$$\forall a\,[q(a) \rightarrow p(a)]$$

is true.

Various logical techniques

Modus ponens and *universal specification* are the basic rules of deduction:

From $p \Rightarrow q$ and p you can conclude q.

From $\forall x\, p(x)$ you can conclude $p(a)$ (where a is the name of some particular thing in the universe of discourse).

Existential generalization: From $p(a)$ you can conclude $\exists x\, p(x)$.

Reasoning by *cases* and by *reductio ad absurdum* require introducing temporary assumptions, which in Greenberg's style of proof writing would be labeled "by hypothesis" [item (1) of Logic Rule 1 (p. 56)]. Eventually one closes off this subdeduction by reaching either a contradiction (refuting the hypothesis) or the desired conclusion (so that you can turn to considering the next case).

Similarly, if you have assumed p and consequently exhibited a proof of q, then you can conclude $p \Rightarrow q$. This is a deduction converse to modus ponens. The universal and existential rules also have converse procedures, but it can be treacherous to use them together: see Part 2 of my logic notes. These last two techniques involve temporarily giving a name to a variable that ultimately belongs inside a quantifier, so that the intermediate steps can be carried out without writing lots of quantifiers.

Substitution rule (p. 66): If $a = b$ and $p(a)$ is a theorem, then $p(b)$ is a theorem.

Incidence geometry (Follow pp. 69–71)

Undefined terms: point; line; incidence (P $\mathtt{I}$ l, stated as "P lies on l" or "l passes through P").

Axioms I-1–I-3, in verbal and symbolic versions (p. 70 and p. 69). Write verbal forms with "nop" and "nil" for "point" and "line".

Defined terms: collinear; concurrent; parallel (write out "parallel" symbolically

$$l \neq m \wedge \neg \exists P (P \mathtt{I} l \wedge P \mathtt{I} m)$$

and note that one clause is redundant by I-2). [Might want to mention parallelism properties here.]

Proposition 2.1: $\forall l \forall m \big[\big(l \neq m \wedge \neg (l \| m) \big) \Rightarrow \exists ! P (P \mathtt{I} l \wedge P \mathtt{I} m) \big]$

Informal argument: Suppose to the contrary that l and m have two (or more) nops in common, P and Q. By I-1, there is a unique nil through P and Q. Then l and m must be the same nil.

Symbolic proof: Expand the hypothesis:

$$l \neq m \wedge \exists P (P \mathtt{I} l \wedge P \mathtt{I} m). \qquad [3][1]$$

Expand the conclusion:

$$\exists P [P \mathtt{I} l \wedge P \mathtt{I} m \wedge \forall Q (Q \mathtt{I} l \wedge Q \mathtt{I} m \Rightarrow Q = P)].$$

(If the P in the hypothesis can be used as the P in the conclusion, we are half done!) Expand I-1:

$$P \neq Q \Rightarrow \exists l [P \mathtt{I} l \wedge Q \mathtt{I} l \wedge \forall m (P \mathtt{I} m \wedge Q \mathtt{I} m \Rightarrow m = l)].$$

Assume we have another Q: $Q \mathtt{I} l \wedge Q \mathtt{I} m$ [2a]; try to prove Q = P. Assume $P \neq Q$ [2b]; then by I-1

$$\exists n [P \mathtt{I} n \wedge Q \mathtt{I} n \wedge \forall k (P \mathtt{I} k \wedge Q \mathtt{I} k \Rightarrow k = n)]. \qquad [4]$$

But we know $P \mathtt{I} l$ (main hyp) and $Q \mathtt{I} l$ (midlevel hyp); so $l = n$. Similarly, $m = n$. So $l = m$. But $l \neq m$ (main hyp); so the inner hyp, $P \neq Q$, is false; so P = Q (given the midlevel hypothesis). That is, we have proved for arbitrary Q that

$$Q \mathtt{I} l \wedge Q \mathtt{I} m \Rightarrow Q = P. \qquad [5]$$

That finishes the proof of the conclusion.

Now compare with the author's semiformal proof on p. 58. There are some minor differences, but we can identify each of our steps with his (bracketed numbers). I tend to write justifications *before* statements (to expound strategy). On homework for Prop. 2.2–5, write either a Greenberg-style formal proof or a symbolic proof, whichever you find easier.

Discuss 2.3 and "useful lemma" below in class. (Beware of prematurely identifying the points and lines in I-3 with those in the proposition.) Note: RAA is more useful for *discovering* proofs than for *writing* proofs.

Useful lemma: For every point P, there exist two other points that are not collinear with P. (Propositions 2.4 and 2.5 will quickly follow.)

Parallelism properties

These definitions (crucial for Exercise 2.9) are hidden in the section "Models".

Elliptic: No parallel lines exist, ever!

Euclidean: $\forall l \forall \mathrm{P}$ not on l there is a unique line m through P parallel to l.

Hyperbolic: $\forall l \forall \mathrm{P}$ not on l there exist many lines (at least 2) through P parallel to l.

IMPORTANT: It is possible that *none of the three* holds. There could be one line and point with a unique parallel and another line and point with many parallels, for instance.

Finite models of incidence geometry

Fig. 2.4: 3 points, 3 lines; elliptic. [Replace lines by Venn ellipses; label points and lines.] ("the triangle")

Fig. 2.5: 4 points, 6 lines; Euclidean. ("the tetrahedron")

Fig. 2.6: 5 points, 10 lines; hyperbolic. (Note that the two dotted lines in back may not intersect, and even if they do, that point is not one of the points of the geometry.)

Dual interpretation of Fig. 2.4: nils = points, nops = lines. [Relabel things consistently.] Can't do that with nonelliptic models. Elliptic models are *self-dual* because of these correspondences, which make the statements symmetrical in "points" and "lines":

I-1 $\leftrightarrow$ elliptic $\parallel \wedge$ 2.1

I-2 $\leftrightarrow$ 2.5

I-3 $\leftrightarrow$ 2.2

2.3 $\leftrightarrow$ 2.4

Philosophical significance of models

We assume our intuition about *finite* sets of points is correct; therefore, verifying the axioms in a finite model shows that the axioms are consistent. We assume that the laws of logic are correct; therefore, the theorems are true in each model. Moreover, we have shown that each of the three $\|$ properties is consistent with the axioms; therefore, *none* of the $\|$ properties can be proved from the I axioms, since then it would be true in all the models! (This doesn't settle the original Euclidean $\|$ debate, because that concerned Euclid's axioms, not the I axioms.)

Remarks on the 3 paragraphs of the "Consistency" section (pp. 76–77)**:**

1. In an inconsistent system, all statements are provable: $(p \wedge \neg p) \Rightarrow q$ is a tautology, by truth tables.
2. This paragraph strikes me as circular and unnecessary. [debate]
3. This one is correct and important.

A quick tour of projective and affine spaces

Definition: An *affine plane* is an incidence geometry with the *Euclidean* $\|$ property.

Definition: A *projective plane* is an incidence geometry with the *elliptic* $\|$ property, and such that every line goes through at least 3 points.

So the "tetrahedron" is an affine plane, but the "triangle" is *not* a projective plane, because it doesn't have enough points.

Theorem: Any affine plane can be extended to a projective plane by adding *points at infinity* and *a line at infinity*. "Parallel lines are declared to meet at infinity." See pp. 82–84 for details.

The projective completion of the tetrahedron is the smallest projective plane, the *Fano plane*, with 7 points and 7 lines (Fig. 2.8).

Projective planes are self-dual.

The real (infinite!) projective plane

1. The ordinary Euclidean plane is affine, and its projective completion is (topologically) a sphere with antipodes identified (Fig. 2.9, Example 7).
2. Let $\mathbf{F}$ be a field. (Think of $\mathbf{F} = \mathbf{R}$.) Then the vector space $\mathbf{F}^2$ is an affine plane whose lines are the loci $ax + by = c$ ("affine subspaces" in linear algebra). The projective plane $\mathbf{P}^2(\mathbf{F})$ can be identified as the set of all *lines through the origin* in $\mathbf{F}^3$. Those

are the nops; the nils are the loci $ax + by + cz = 0$. Note that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \simeq \begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \end{pmatrix} \quad \text{and similarly for} \quad (\, a \quad b \quad c \,)\,.$$

Self-duality is clear. (Note: Lines through the origin are in 1–1 correspondence with points on the unit sphere, antipodally identified.)

If $\mathbf{F}$ is the finite field of two elements, $\{0, 1\}$, then $\mathbf{F}^2$ is the tetrahedron and $\mathbf{P}^2(\mathbf{F})$ is the Fano plane (Major Exercise 5).