Chapter 3

Betweenness (ordering)

"Point B is *between* point A and point C" is a fundamental, undefined concept. It is abbreviated $A * B * C$.

A system satisfying the incidence and betweenness axioms is an *ordered incidence plane* (p. 118).

The first 3 axioms (p. 108)

B-1 is really two statements.

Corollary to B-2: There are infinitely many points.

Note to B-3: The real projective plane is excluded. In fact, all elliptic geometries are excluded, because in Ch. 4, p. 163, we will prove that parallel lines always exist.

Definitions (same as in Ch. 1; p. 109)**:**

- *segment* AB
- *ray* $\overrightarrow{\mathrm{AB}}$
- set of points on a line, $\{\overleftrightarrow{\mathrm{AB}}\}$

Proposition 3.1. [Proof of (i) in book, p. 109; proof of (ii) by a team next time]

Definitions: *same side* and *opposite side* (p. 110). Note that the language is loaded: What is a "side"?

Axiom B-4 (plane separation property) and corollary (pp. 110–111)

Note to B-4: Dimensions higher than 2 are now excluded.

Definition: *side* = *half-plane* (p. 111)

Side discussion (cf. pp. 82–83):

- *equivalence relation* (3 properties)
- *equivalence classes*
- *partition* (converse construction)
- *cosets* (cf. Euclidean lines and planes)
- *factor space* (cf. real projective plane)

Proposition 3.2 (p. 112) (an example of the foregoing)

Proposition 3.3 (pp. 112–113) [prove first part; rest for a team]

Corollary [team]

[Henceforth repeat book theorems only when necessary to filibuster for teams.]

Proposition 3.4 (line separation property) (p. 113–114)

Pasch's Theorem (p. 114)

Proposition 3.5 [team]

Proposition 3.6 [team]

Definition: *interior* for angles (p. 115)

Proposition 3.7 [team]

Proposition 3.8 [team]

Definition: *betweenness* for rays, and the warning above it (p. 115)

Crossbar Theorem (p. 116)

Definitions: *interior* and *exterior* for triangles (p. 117)

Proposition 3.9 [team]

Congruence

Congruence is an undefined term for both segments and angles, and a defined term for triangles.

Axiom C-1 (p. 119) is essentially "Euclid II" (p. 16).

Axioms C-2 and C-5: Congruence of segments and congruence of angles are both equivalence relations.

Remark: Greenberg assumes "twisted transitivity"

$$a \cong b \wedge a \cong c \ \Rightarrow\ b \cong c$$

and reflexivity and proves symmetry. Ordinary, but not twisted, transitivity is satisfied by order relations such as $\leq$.

Axiom C-3: "Equals added to equals" for segments along their respective lines. (The analog for angles will be a theorem (Prop. 3.19). Likewise the analogs for subtraction, Prop. 3.11 and 3.20.)

Axiom C-4: A given angle can be attached to a given ray on either side (but otherwise uniquely).

Axiom C-6: SAS congruence criterion. This famous "theorem" turns out to be independent of the other axioms. (We will prove that in Exercise 35.) Euclid's "proof" uses an intuitive concept of moving figures rigidly that is foreign to his axioms and standard methods.

Proposition 3.10: (p. 123) An isosceles triangle has equal base angles. (Proof in book. Converse in homework, Prop. 3.18.) Note that the Pappus proof and the definition of congruence of triangles regard reflection-symmetric triangles as congruent. Note that in the definition of "angle" on p. 18 we have a *set* of two rays, not an *ordered pair* of rays, so an angle is identical with itself "in reverse".

Proposition 3.12 and **Definitions** of segment ordering (p. 124): See book and use in the following.

Proposition 3.13: (segment ordering, p. 125) Work out in class: Exercises 21–23.

For the next batch of propositions (p. 125) we need to import some definitions from Chapter 1:

supplementary angles

vertical angles (Exercise 1.4)

right angles

perpendicular lines (The passage from rays to lines (some tedious logical bookkeeping) is stated in the exercise section, p. 42.)

Proposition 3.14: Supplements of congruent angles are congruent. (proved in exercise section)

Proposition 3.15: [prove in lecture]

(a) Vertical angles are congruent to each other.

(b) An angle congruent to a right angle is a right angle.

Proposition 3.16: For every line l and every point P there is a line through P perpendicular to l. [proved in book; go through it] Uniqueness is not yet clear.

Propositions 3.17 and 3.22: ASA and SSS. (homework)

Proposition 3.23: (p. 128) "Euclid IV" — All right angles are congruent. [go through proof if time permits]

Ordering for angles (and Euclid IV) allows *acute* and *obtuse* to be defined.

A *Hilbert plane* is a model satisfying all the I, B, and C axioms.

[Work out Exercise 35 (congruence part) to show independence of SAS.]

Continuity

Greenberg states a large number of rival continuity axioms, but he prefers two:

Circle-Circle Continuity Principle: If circle γ has one point inside and one point outside circle γ', then the two circles intersect in two points.

Dedekind's Axiom: If $\{l\}$ is a disjoint union $\Sigma_1 \cup \Sigma_2$ with no point of one subset between two points of the other (and neither subset empty), then there exists a unique O on l such that either Σ_1 or Σ_2 is a ray of l with vertex O (and then the other subset is the opposite ray with O omitted). (I.e., each line is (at least locally) like the real line $\mathbf{R}$ as Dedekind defined the latter. "At least locally" means that the geometrical line may be modeled by an interval of $\mathbf{R}$, not all of $\mathbf{R}$.)

Dedekind's assumption is very strong; circle-circle is fairly weak (and implied by Dedekind). Some kind of continuity assumption is necessary to guarantee that the would-be intersection points of circles with lines, segments, or other circles are not accidentally "holes" in the space. For example, the *rational* plane $\mathbf{Q}^2$ lacks points with irrational coordinates, so "$y = \sqrt{1-x^2}$" may be problematical.

We do not commit to any one continuity axiom.

Hilbertian parallelism

Hilbert's Euclidean Axiom of Parallelism: For every l and every P not on l, there is *at most* one line though P and parallel to l.

This seemingly leaves open the possibility that there are *no* parallels for the given l and P. (Recall that if there are *never* parallels, the geometry would be classed as elliptic.) However, it turns out that the other Hilbert axioms already rule out this possibility: Hilbert geometries are either Euclidean or hyperbolic. (An axiomatic development of elliptic geometry requires different axioms; see Appendix A.)

Definitions: A *Euclidean plane* is a Hilbert plane satisfying Hilbert parallelism and circle-circle continuity. A *real Euclidean plane* is a Hilbert plane satisfying Hilbert parallelism and Dedekind continuity.

Not surprisingly, $\mathbf{R}^2$ is a real Euclidean plane (essentially the only one). As mentioned, $\mathbf{Q}^2$ is not even a (circle-circle) Euclidean plane. A non-Dedekind Euclidean plane is described on pp. 140–141. (You need enough algebraic numbers to take square roots.)