Chapter 4

Outline of chapter

1. More standard geometry (interior and exterior angles, etc.)
2. Measurement (degrees and centimeters)
3. Statements equivalent to the parallel postulate
4. Saccheri and Lambert quadrilaterals (crucial for later chapters)

Additional neutral geometry

Proofs are in the book or will be assigned to future presentations by a team.

Assume a Hilbert plane (Axioms I, B, C). Continuity axioms will be specified when needed.

Theorem 4.1 (AIA): Given two lines cut by a third line, if a pair of alternate interior angles are congruent, then the two lines are parallel. [figure to clarify terminology]

Remarks: The converse is not true in hyperbolic geometry. The theorem itself is not true in elliptic geometry (sphere or projective plane).

Proof [do!] uses B-4 (plane separation) and I-1 (uniqueness of the line through 2 points, in the form of its dual, Prop. 2.1).

Corollary 1: (a) Two lines perpendicular to the same line are parallel. (b) The perpendicular from P to l is unique.

Corollary 2: Given l and P not on l, there exists at least one line through P parallel to l.

Remark: This is a striking result! It says that "elliptic parallelism" is inconsistent with the Hilbert axioms. There are two natural candidates for elliptic geometry: the sphere (with the great circles as the lines) and the projective plane (the sphere with antipodal points identified). Since two great circles intersect at their two "poles", the sphere violates I-1. The projective plane restores I-1 but violates B-4: a line no longer has two separate sides. Also, both proposed models (interpretations) violate B-3: 3 collinear points lie on a circle, so none of them can be distinguished as "between" the other two.

Theorem 4.2 (EA): An exterior angle of a triangle (supplementary to one of its interior angles) is greater than either remote interior angle.

Corollary 1: If one angle is right or obtuse, the other two are acute.

Prop. 4.1: SAA congruence criterion.

Prop. 4.2 (hypotenuse-leg criterion): Two right triangles are congruent if the hypotenuse and one leg of one are respectively congruent to those of the other.

Now we recover some of the things we reviewed from edge-and-compass constructions in Chap. 1:

Prop. 4.3 Every segment has a unique midpoint.

Prop. 4.4 (a) An angle has a unique bisector. (b) A segment has a unique perpendicular bisector.

Prop. 4.5: In a triangle the greater angle lies opposite the greater side and the greater side lies opposite the greater angle.

(This settles a point that bedeviled us in trying to prove SSS.)

Prop. 4.6: Given two triangles (primed and unprimed), if $\mathrm{AB} \cong \mathrm{A'B'}$ and $\mathrm{BC} \cong \mathrm{B'C'}$, then angle $\mathrm{B} <$ angle $\mathrm{B'} \iff \mathrm{AC} < \mathrm{A'C'}$.

Measurement (rulers and protractors at last!)

Why were the Greek geometers so leery of assigning numbers to lengths and angles? The stock explanation (probably further oversimplified by me) is that they had no concept of the real-number continuum; in the early days they tried to think of lines as made up of evenly spaced discrete points, but Zeno's paradoxes and the irrationality of $\sqrt{2}$ convinced them that their concepts of measurement were hopelessly inconsistent. The most they were willing to countenance was statements like "segment CD is congruent to the sum of five segments congruent to AB" or " . . . to the sum of five segments congruent to AB and one segment smaller than AB." The Axiom of Archimedes states that two segments can always be compared in this way. Nowadays we would choose one favorite segment and define it to be the *unit of length* to be used in all such comparisons. The situation for angles is similar, except that there is a natural unit, the right angle, which we arbitrarily divide by 90 (thanks to the Babylonians) for convenience.

Remark: All such historical remarks found in books or lectures by nonhistorians should be regarded with skepticism. For one thing, Greek geometry extended from Thales (c. 600 BC) to Hypatia and Proclus (after 400 AD) — 1000 years. So saying "The Greeks believed . . . " is like saying "The French believed . . . " without specifying whether you're talking about Charlemagne or Marie Curie.

Measurement Theorem 4.3 (too long to copy, p. 170) describes the success of this program. The theorem assumed the Archimedes axiom; Dedekind's (stronger) is needed for two parts, the existence of lengths and angles corresponding to each real number.

Complementary angles can now be defined in the usual way. (Why are Props. 3.19 and 3.20 not enough? See Ex. 33(a).)

Corollary 2: The sum of any two angles of a triangle is less than 180°.

Triangle Inequality: Any side of a triangle is less [in length] than the sum of the lengths of the other two sides.

Remark: To state this theorem, Euclid had to construct a segment with pieces congruent to the two sides.

Corollary (converse to the triangle inequality): Circle-circle continuity $\iff$ Given 3 lengths with the sum of any two greater than the third, there exists a triangle whose sides have those lengths.

Propositions equivalent to the Euclidean parallel postulate

Recall:

Hilbert's Euclidean parallel postulate (HE): For every l and every P not on l, there is at most one line through P and parallel to l (and hence *exactly* one, by Corollary 2 to AIA).

Euclid's original fifth postulate (EV): Given two lines and a transversal, if the two interior angles on one side add to less than a right angle, then the two lines intersect on that side of the transversal.

Remark: Greenberg formulates EV and the next proof in terms of degree measure, presumably to emphasize that angle addition in purely Euclidean terms requires a step of angle transplantation by C-4 and Prop. 3.19.

Theorem 4.4: HE $\iff$ EV.

[Go through proof from book.]

Prop. 4.7: HE $\iff$ If a line intersects one of two parallel lines, then it intersects the other.

Prop. 4.8: HE $\iff$ converse of AIA.

Prop. 4.9: HE $\iff$ If t is a transversal to parallels l and m, and $t \perp l$, then $t \perp m$.

Prop. 4.10: HE $\iff$ If $k \parallel l$, $m \perp k$, and $n \perp l$, then either $m = n$ or $m \parallel n$.

Note that the following two are *one-way* implications:

Prop. 4.11: HE $\Rightarrow$ The three angles of a triangle sum to 180°.

[Go through proof from book.]

Corollary: HE $\Rightarrow$ An exterior angle of a triangle is equal to the sum of the remote interior angles.

Saccheri and Lambert quadrilaterals

Def. Q (p. 44)**:** Given four points A, B, C, D such that

(i) no three of the points are collinear, and

(ii) any pair of the specific segments AB, BC, CD, and DA either have no point in common or have only an endpoint in common,

the *quadrilateral* □ABCD "consists of" these four segments.

Remarks:

1. Greenberg doesn't make clear whether a quadrilateral is a set of four segments or a set of points, the union of those segments. It makes no difference conceptually, but would make technical differences in the truth or falsity of certain statements (much as in the homework question "Is a triangle convex?"). Let's say it is the set of segments.

2. The order of the points matters! The next definition makes it matter even more.

Definitions S: □ABDC is *bi-right* if angle A and angle B (which must be adjacent!) are both right angles; it is *isosceles* if AC $\cong$ BD; it is a *Saccheri* quadrilateral if both bi-right and isosceles. AB is called the *base*, CD is called the *summit*, AC and BD are called the *sides*, and angles C and D are called *summit angles*.

Def. L: A quadrilateral with (at least) 3 right angles is a *Lambert quadrilateral*. The "fourth angle" means whichever angle is not already known to be right by hypothesis.

Prop. 4.12: (a) The summit angles of a Sac. quad. are congruent. (b) The line joining the midpoints of the summit and the base is perpendicular to both those segments.

Corollary: (See Fig. 4.15.) The line in 4.12(b) divides the Sac. quad. into two Lam. quads. with fourth angle equal to the summit angle. Conversely, a Lam. quad. can be *doubled* to form a Sac. quad. by reflecting it through either of the sides that are not part of the 4th angle.

The proof of 4.12 is a routine exercise in congruent triangles. It would be very surprising if the theorem were not true, since that would indicate an asymmetry of space.

Theorem: (See Fig. 4.19.) HE $\Rightarrow$ all Lam. quads. and Sac. quads. are rectangles!

Prop. 4.13: In any bi-right quad., the greater side is opposite the greater angle.

[Important proof, pp. 178–179. The RAA can be replaced by a trichotomy argument.]

Corollaries 1 and 2: (See Figs. 4.17–18, pp. 179–180.) Perpendicular segments from one side of an acute angle to the other increase as one moves away from the vertex. Thus EV $\Rightarrow$ Aristotle's Axiom (p. 133), because then the starting segment can be chosen arbitrarily long. (I.e., the segments increase *unboundedly*.)

Remark: When Greenberg says that Aristotle's axiom is a consequence of Dedekind's or Archimedes's axiom, he means a consequence of that *and the other Hilbert axioms*. In my opinion Aristotle is not really a "continuity axiom", although it requires some continuity principle for its proof (which shows up in Ch. 5, Exercise 2). It is one of the many principles that exclude the standard elliptic geometries. Note that it is consistent with hyperbolic parallelism, so Aristotle does not imply EV.

Corollary 3: A side adjacent to the fourth angle θ of a Lam. quad. is $\left\{\begin{matrix} > \\ \cong \\ < \end{matrix}\right\}$ its opposite side when θ is $\left\{\begin{matrix} \text{acute} \\ \text{right} \\ \text{obtuse} \end{matrix}\right\}$.

Corollary 4: The summit of a Sac. quad. is $\left\{\begin{matrix} > \\ \cong \\ < \end{matrix}\right\}$ the base when the summit angle is $\left\{\begin{matrix} \text{acute} \\ \text{right} \\ \text{obtuse} \end{matrix}\right\}$.

Uniformity theorem: If *one* Sac. quad. has acute summit angles, then so do *all*. Same for right and obtuse, and for fourth angles of Lam. quads.

The proof occupies 4 major exercises. I will endorse consortia of 4 people to do these as their second papers. Rules: **Each of the four papers must have a (distinct) primary author who takes responsibility for writing that paper.** Some joint work on the mathematical content is expected. Each paper should start with the label "Part of a joint project with [the other three names]."

Definition: A Hilbert plane is *semi-Euclidean* if all Lam. (and Sac.) quads. are rectangles. If the fourth Lam. angle is acute, we say the plane satisfies the *acute angle hypothesis* (and similarly for *obtuse*).

Corollary: Rectangles exist iff the plane is semi-Euclidean; opposite sides are always congruent.

More corollaries pp. 182–183 just repeat previous propositions in the uniformed contexts.

Saccheri's Angle Theorem: In a Hilbert plane,

(a) $\exists\,\Delta$: angle sum $< 180^\circ \;\Rightarrow\; \forall\,\Delta$: angle sum $< 180^\circ \iff$ acute angle hyp.

(b) Same for $= 180^\circ$ and semi-Euclidean.

(c) Same for $> 180^\circ$ and obtuse.

Saccheri–Legendre Theorem: Archimedes's Axiom $\Rightarrow$ angle sums $\leq 180°$ (hence refuting the obtuse angle hypothesis).

(This is no big surprise, since we expect the obtuse angle case to correspond to elliptic geometry, which we already know to be inconsistent with the Hilbert axioms.)