Chapters 5 and 6

I won't repeat most of the history. That doesn't mean it's unimportant, only that I don't have anything to add.

Pp. 248–249: "A model of hyperbolic plane geometry is a sphere of imaginary radius with antipodal points identified in the three-dimensional spacetime of special relativity." This is better explained on pp. 311–313. I will come back to it later.

RESULTS IN NON-EUCLIDEAN HILBERT GEOMETRY (pp. 250ff)

Negation of HE: ∃ a line l and a P not on l such that at least two distinct lines through P are $\parallel$ to l.

Def.: A Hilbert plane satisfying this is *non-Euclidean.*

Basic Theorem 6.1: A non-Euclidean Hilbert plane satisfying Aristotle's axiom satisfies the acute angle hypothesis (and hence a list of things we proved in the last chapter).

Remark: Recall that "rectangles do not exist" is a quick way of saying that either the acute angle hypothesis or the obtuse angle hypothesis holds. (See p. 182.)

Universal Non-Euclidean Theorem and Corollary: In a Hilbert plane in which rectangles do not exist, for every l and every P not on l there are at least 2 parallels to l through P (and hence infinitely many).

Def.: The *defect* of a triangle is the amount by which the three angles fail to total $180°$.

Prop. 6.1: If a triangle is divided into 2 triangles, its defect is the sum of the defects of the parts.

Remark: In Chap. 10 it is proved that the *area* of a triangle is proportional to its defect. (The constant of proportionality is fixed by the scale factor of the hyperbolic geometry — the "radius" R of the hyperboloid in the model described above.) It follows that in any particular hyperbolic geometry there is an upper limit on the area of triangles, since the defect can't be more than $180°$. The next proposition shows how this can be true, even though segments can be arbitrarily long:

Prop. 6.2: Acute hyp. $\Rightarrow$ if two triangles are similar, then they are congruent. (AAA is true!) [sketch]

Related remark: HE $\iff$ **Wallis's Axiom:** Given any segment and any triangle, there is a similar triangle built on that segment.

Prop. 6.3: In a plane in which rectangles do not exist, parallel lines are not equidistant. In fact, any set of point on l equidistant from l' contains at most two points.

Related remark: semi-Euclidean $\Longleftrightarrow$ **Clavius's Axiom:** The parallel through P is the equidistant locus through P (both relative to a line l).

Remark: "I hope the reader is not too shocked to see line l drawn as being 'curved'!" This should not be a shock to those who have drawn meridians on a sphere. [2 sketches]

Limiting parallel rays

See Fig. 6.10 (p. 258), and Fig. 6/14 (p. 264), and Figs. 7.9–10 (p. 304). Given l and P, some lines through P (those closest to the perpendicular) intersect l, others do not. Is there a boundary case between those that do and those that don't? Under sufficient continuity axioms, it can be proved that a boundary ray does exist and it does not intersect. In other words, there is a first ray that fails to meet l. Cf. least upper bound vs. maximum, and the possible nonexistence of either if $\mathbf{R}$ is replaced by $\mathbf{Q}$. (Greenberg's theorem uses Aristotle and line-circle; Dedekind would do for both.)

Hilbert made this property another axiom:

Hilbert's Hyperbolic Axiom of Parallels: $\forall\, l$, P, a limiting parallel ray exists, and it is not $\perp$ to the $\perp$ from P to l.

Definitions: A Hilbert plane obeying this axiom is a *hyperbolic plane*. A non-Euclidean plane satisfying Dedekind's axiom is a *real hyperbolic plane*.

$$\text{real hyperbolic} \Rightarrow \text{hyperbolic} \Rightarrow \text{non-Euclidean}$$

Corollaries: A Hilbert plane satisfying Dedekind's axiom is either real Euclidean or real hyperbolic. A Hilbert plane satisfying Aristotle's axiom and line-circle continuity is either Euclidean or hyperbolic.

Theorem 6.3: In a hyperbolic plane, if m is $\parallel$ to l, then *either* m contains a limiting parallel ray in one direction or the other, *or* there is a (unique) common $\perp$ to l and m (but not *both*). (See Fig. 6.14, p. 264.)

Prop. 6.6: The angle between the perpendicular and the limiting ray is acute and is the same on both sides. (Its measure is called the *angle of parallelism* associated with that $\perp$ segment.)

Dehn's models

[Table from Dehn's 1900 paper (table translated by G. B. Halsted)]

This situation makes necessary the complicated statement of hypotheses in the theorems about Saccheri–Lambert quadrilaterals.

See pp. 188-189 and 250, especially footnote on p. 189. My attempt to elaborate: (These remarks should be considered intuitive only.)

Start by noting that $\mathbf{F}^2$ and $\mathbf{F}^3$ model Euclidean 2D and 3D geometry (Theorem, p. 141).

There are fields containing "infinitesimal" elements, say a, such that

$$na \equiv \sum_{j=1}^{n} a < r \quad \text{for any } r \in \mathbf{Q}^{+}.$$

(Thus Archimedes's axiom fails for segments of such lengths.) The modern theory of *nonstandard analysis* attempts (among other things) to vindicate pre-19th-century calculus in this way (i.e., making equations like $dy = \frac{dy}{dx} dx$ literally true); it developed in the mid-20th century, hence 50 years after Dehn, but Dehn already knew about non-Archimedean fields. The apparent judgment of history is that calculus doesn't need this: the rigorous theory of limits is better. $\mathbf{Q}$ and $\mathbf{K}$ are too small; fields $\mathbf{F}$ with infinitesimals are two big; $\mathbf{R}$ is just right.

Nevertheless, a rigorous theory of infinitesimals allows one to create models of non-Legendrean and semi-Euclidean geometry. Let Π be the subset of $\mathbf{F}^2$ of points whose coordinates are infinitesimal. Since you can't get out of Π by adding elements of Π, the Hilbert axioms are still satisfied, so Π is also a model for them. Also, it satifies the right angle hypothesis because $\mathbf{F}^2$ does. However, two lines in $\mathbf{F}^2$ determined by pairs of points in Π that intersect in a point not in Π count as parallel in Π. Therefore, there are lots of parallels; HE/EV fails although the plane is semi-Euclidean (rectangles exist; triangles add to $180°$).

Next, consider a sphere in $\mathbf{F}^3$. (This is not precisely Dehn's construction, but seems to be consistent with Greenberg's footnote.) Consider an infinitesimal neighborhood of a point on the sphere. For example, introduce polar coordinates (θ, ϕ) around the north pole and restrict θ to infinitesimal values (getting a very tiny polar cap). Now most pairs of great circles do not intersect inside the cap, so they are parallel. However, the fourth angles of Lambert quads. inside the cap are obtuse (although only *infinitesimally* larger than a right angle).