Math. 467 (Fulling) 15 Dec 2009

Final Examination – Solutions

Name: ______________________________

1. *(40 pts.)* was the take-home essay.

2. *(Multiple choice – each 5 pts.)*

   (a) Which of these is a theorem in neutral geometry?

   (A) Given any triangle ΔABC and any segment DE, there exists a triangle ΔDEF (having DE as one of its sides) that is similar to ΔABC.

   (B) If two lines cut by a transversal l have a pair of congruent alternate interior angles with respect to l, then the two lines are parallel.

   (C) If two parallel lines are cut by a transversal, then the resulting alternate interior angles are congruent.

   (D) For any line l and any point P not on l, the curve through P of points that are equidistant from l is a straight line parallel to l.

B (AIA theorem). The other three (Wallis's axiom, converse to AIA, Clavius's axiom) are all equivalent to the Euclidean parallel postulate.

   (b) The ratio of a circle's circumference to its radius is

   (A) less than 2π in non-Euclidean geometry and equal to 2π in Euclidean geometry.

   (B) greater than 2π in elliptic geometry and less than 2π in hyperbolic geometry.

   (C) less than 2π in elliptic geometry and greater than 2π in hyperbolic geometry.

   (D) always equal to 2π.

C

   (c) We needed to prove the alternate interior angle theorem and exterior angle theorem before proving the congruence criterion

   (A) SSS

   (B) ASA

   (C) SAA

   (D) SAS

C

   (d) Poincaré's disk model is better than Klein's in which respect?

   (A) It satisfies more of the axioms.

   (B) It is *conformal*, meaning that it represents angles accurately.

   (C) Its lines are straight Euclidean lines.

   (D) [none of these]

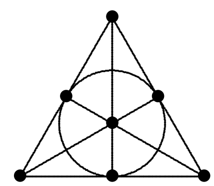
B

(e) Similar triangles are congruent
(A) always.
(B) never: this is the notoriously fallacious "AAA" congruence criterion.
(C) in hyperbolic geometry.
(D) only in Dehn models built on non-Archimedean fields.

C

(f) In this course, a *right angle* is **defined** as one that
(A) is congruent to its supplement.
(B) contains 180°.
(C) is a base angle of a Saccheri quadrilateral.
(D) is the foot of a perpendicular line.

A

(g) *(Bonus question)* Which statement is false?  This plane with 7 points and 7 lines
(A) is called the *Fano plane*.
(B) is the smallest projective plane.
(C) satisfies the Hilbert incidence axioms.
(D) has the Euclidean parallel property.

D (It has the *elliptic* parallel property — all lines intersect.)

3. *(25 pts.)* Prove (within neutral geometry) the *hypotenuse–leg theorem*: Two right triangles are congruent if the hypotenuse and one other side of one triangle are congruent (respectively) to the hypotenuse and a side of the other triangle.

[See Ex. 4.4, p. 193. Note that it is not enough to move the two triangles together and declare the resulting figure an isosceles triangle; you must prove that the bottom side is a straight line, either by constructing it that way (the hint in the book) or by remarking that right angles are supplementary (so two of them adjacent must share a line).]

4. *(Essay – 10 pts.)* Explain why this statement is false:
Although 2000 years of efforts to prove the parallel postulate as a theorem in neutral geometry have been unsuccessful, it is still possible that someday some genius will succeed in proving it.

[A magic word is "model".]

5. *(20 pts.)* Rearrange these names into historical order, earliest to latest:

Saccheri, Euclid, Lobachevsky, Hilbert, Thales

Thales, Euclid, Saccheri, Lobachevsky, Hilbert

6. *(25 pts.)* Do **ONE** of these: (Use blank sheet.)
(NO extra credit for doing both. Indicate which one you want graded!)

(A) In the Poincaré disk model, the formula for arc length is

$$ds^2 = \frac{4(dx^2 + dy^2)}{[1 - (x^2 + y^2)]^2}\,.$$

Introduce polar coordinates (ρ, θ) in the (x, y) plane and show that the further coordinate transformation

$$\rho = \tanh\left(\frac{r}{2}\right)$$

converts the arc length to

$$ds^2 = dr^2 + \sinh^2 r\, d\theta^2\,.$$

Explain why this result justifies the vague claim that hyperbolic geometry describes "a sphere of imaginary radius".

[See Chapter 7 notes on the Web page, pp. 38–39. However, the calculation is greatly simplified because here you are given the transformation and just need to verify that it does the right thing; there is no need to introduce the unknown function f.]

(B) Suppose that A and B are points in the Poincaré disk and that P and Q (points on the circle bounding the disk) are the ends of the Poincaré line through A and B. The *cross-ratio* is defined as

$$(\mathrm{AB}, \mathrm{PQ}) \equiv \frac{\overline{\mathrm{AP}}}{\overline{\mathrm{AQ}}}\,\frac{\overline{\mathrm{BQ}}}{\overline{\mathrm{BP}}}\,,$$

where, for instance, $\overline{\mathrm{AB}}$ is the *Euclidean* distance between A and B. Show how to use the cross-ratio to define a distance (length) function in the hyperbolic geometry that is *additive* in the sense that

$$d(\mathrm{AB}) = d(\mathrm{AC}) + d(\mathrm{CB})$$

when $\mathrm{A} * \mathrm{C} * \mathrm{B}$ along a Poincaré line. (Verify that it is indeed additive.)

[See pp. 319–321.]