Group Zeta (Garcia, Hine, Maciejeski, Rohan)
MATH 467
Exercise 2 (b), (d)

Proposition 3.3 Given A * B * C and A * C * D. Then B * C * D and A * B * D (see Figure 3.9).

PROOF:
PART 1:: Proof of B * C * D
(Note: This is copied from Greenberg, but has a clarification on the justification of step 3.)

(1) A, B, C, and D are four distinct collinear points (see Exercise 1).
(2) There exists a point E not on the line through A, B, C, D (Proposition 2.3).
(3) Consider line EC. Since (by PART 1 step (1)) AD meets this line in point C, points A and D are on opposite sides of EC.
(4) We claim A and B are on the same side of EC. Assume on the contrary that A and B are on opposite sides of EC (RAA hypotheses).
(5) Then EC meets AB in a point between A and B (definition of "opposite sides").
(6) That point must be C (Proposition 2.1).
(7) Thus, A * C * B but we are given A * B * C, which contradicts Betweenness Axiom 3.
(8) Hence, A and B are on the same side of EC (RAA conclusion).
(9) B and D are on opposite sides of EC (steps 3 and 8 and the corollary to Betweenness Axiom 4).
(10) Hence, the point C of intersection of lines EC and CD lies between B and D (definition of "opposite sides"; Proposition 2.1, i.e., that the point of intersection is unique).
(11) Therefore, B * C * D.

PART 2:: Proof of A * B * D

(1) A, B, C, and D are four distinct collinear points (see Exercise 1).
(2) There exists a point E not on the line through A, B, C, D (Proposition 2.3).
(3) Consider line EB. Since (by PART 2 step (1)) AC meets this line in point B, points A and C are on opposite sides of EB.
(4) We claim C and D are on the same side of EB. Assume on the contrary that C and D are on opposite sides of EB (RAA hypotheses).
(5) Then EB meets CD in a point between C and D (definition of "opposite sides").
(6) That point must be B (Proposition 2.1).
(7) Thus, C * B * D but from PART 1 we have B * C * D, which contradicts Betweenness Axiom 3.
(8) Hence, C and D are on the same side of EB (RAA conclusion).
(9) A and D are on opposite sides of EB (steps 3 and 8 and the corollary to Betweenness Axiom 4).
(10) Hence, the point B of intersection of lines EB and AC lies between A and D (definition of "opposite sides"; Proposition 2.1, i.e., that the point of intersection is unique).
(11) Therefore, A * B * D.

COROLLARY. Given A * B * C and B * C * D. Then A * B * D and A * C * D.

PROOF:

PART 3:: Proof of A * C * D

(1) By Betweenness Axiom 1, if A * B* C, then A, B, and C are three distinct collinear points, and if B * C * D, then B, C, and D are distinct collinear points.
(2) Assume A=D.
(3) Thus, D * B * C but we are given B * C * D, which contradicts Betweenness Axiom 3.
(4) Hence, A≠D, and A, B, C, and D are four distinct points.
(5) By Incidence Axiom 1, B and C uniquely determine a line, let's say *l*.
(6) By PART 3 step (1), A lies on the same line *l* as B and C, and D lies on the same line as B and C. So A, B, C, and D are distinct collinear points.
(7) There exists a point E not on the line through A, B, C, D (Proposition 2.3).
(8) Consider line EC. Since (by PART 4 step (1)) BD meets this line in point C, points B and D are on opposite sides of EC.
(9) We claim A and B are on the same side of EC. Assume on the contrary that A and B are on opposite sides of EC (RAA hypotheses).
(10) Then EC meets AB in a point between A and B (definition of "opposite sides").
(11) That point must be C (Proposition 2.1).
(12) Thus, A * C * B but we are given A * B * C, which contradicts Betweenness Axiom 3.
(13) Hence, A and B are on the same side of EC (RAA conclusion).
(14) A and D are on opposite sides of EC (steps 3 and 8 and the corollary to Betweenness Axiom 4).
(15) Hence, the point C of intersection of lines EC and AD lies between A and D (definition of "opposite sides"; Proposition 2.1, i.e., that the point of intersection is unique).
(16) Therefore, A * C * D.

PART 4:: Proof of A * B * D

(1) By PART 3 step (4 and 6), A, B, C, and D are distinct collinear points.
(2) Since A * B * C is given and Part 3 proves A * C * D, then by Proposition 3.3, A * B * D.