ΣΥΓΓΡΑΜΜΑ Β′

'ΕΝΩΜΟΤΙΑ Α′
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Theorem 1 ***Uniqueness of the midpoint.*** *Suppose there were a segment, $\overline{AB}$, with two midpoints, M and M'.*

It must be that $M = M'$.

Proof. Note that, because $A * M * B$ and $A * M' * B$, Axiom B-3 implies that either $A*M*M'$, $A*M'*M$, or $M = M'$. However, because it is possible to relabel points at will, the only two cases we need to consider are: $A * M * M'$ and $M = M'$.

Suppose $A * M * M'$. By the definition of the "greater than" sign (in relation to segments), we may state: $AM < AM'$. By the definition of midpoint, $AM = MB$ and $AM' = M'B$. Thus $MB < M'B$. However $M * M' * B$, so $CB > DB$. This is a contradiction.

Because the only other case causes a contradiction, it must hold that $M = M'$. ■

Theorem 2 ***An angle has a unique bisecting ray.*** *Suppose there were an angle, $\angle POQ$. There exists a unique ray, $\overrightarrow{OM}$, such that $\angle POM \cong \angle MOQ$.*

Proof. Consider the points A und B, such that A lies upon $\overrightarrow{OP}$, B lies upon $\overrightarrow{OQ}$, and $\overline{OA} \cong \overline{OB}$.

Now, consider the midpoint, M of $\overline{AB}$. Because M is the midpoint, $\overline{MA} \cong MB$. However $\overline{OM} \cong \overline{OM}$ and $\overline{OA} \cong \overline{OB}$. Thus by Side-Side-Side congruence, it must be that $\triangle MOA \cong \triangle MOB$, and $\angle MOA \cong \angle MOB$.

Also, observe that the uniqueness of the midpoint guarantees the uniqueness of the bisector. ■

Theorem 3 ***A segment has a unique perpendicular bisector.*** *Suppose there were a segment $\overline{AB}$. There is a unique perpendicular bisector, $\overleftrightarrow{\ell}$ to the segment.*

Proof. Consider the midpoint of the segment, M. Also consider a point P such that $\overline{AP} \cong \overline{BP}$. Now, $\overline{PM} \cong \overline{PM}$, $\overline{AM} \cong \overline{BM}$, and $\overline{AP} \cong \overline{BP}$. This means that $\triangle AMP \cong \triangle BMP$.

This congruence implies that $\angle AMP \cong \angle BMP$, which, as $A * M * B$, implies $\angle AMP$ and $\angle BMP$ are right angles.

Therefore, $\overleftrightarrow{PM}$ goes through the midpoint of $\overline{AB}$ and $\overleftrightarrow{PM} \perp \overleftrightarrow{AB}$. ■