Math. 467 (Fulling) 16 October 2009

Midterm Test – Solutions

Name: ________________________________

1. *(10 pts.)* All but two of the following propositions are theorems of Hilbert geometry (that is, they can be proved from the I, B, and C axioms without additional assumptions). Identify the two statements that don't belong.

   (A) If $\mathrm{A} * \mathrm{B} * \mathrm{C}$, then B is the only point common to rays $\overrightarrow{\mathrm{BA}}$ and $\overrightarrow{\mathrm{BC}}$.

   (B) If A lies on the line l and B does not, then every point of $\overrightarrow{\mathrm{AB}}$ except A is on the same side of l as B.

   (C) If D lies in the interior of angle $\sphericalangle\mathrm{BAC}$, then there exist points E on $\overrightarrow{\mathrm{AB}}$ and F on $\overrightarrow{\mathrm{AC}}$ such that $\mathrm{E} * \mathrm{D} * \mathrm{F}$.

   (D) If $\overrightarrow{\mathrm{AD}}$ is between $\overrightarrow{\mathrm{AC}}$ and $\overrightarrow{\mathrm{AB}}$, then $\overrightarrow{\mathrm{AD}}$ intersects segment BC.

   (E) If a ray r emanating from an exterior point of triangle ABC intersects side AB in a point between A and B, then r also intersects side AC or side BC.

   (F) If a ray r emanates from an interior point of triangle ABC, then r intersects exactly one of the sides of that triangle.

C and F don't belong.

C is not a theorem of Hilbert geometry and is false in hyperbolic geometry; see "Warning", p. 115.

F is false; the correct statement is Prop. 3.9(b).

A is part of Prop. 3.6; B is the oft-used lemma Exercise 3.9; D is the crossbar theorem; E is Prop. 3.9(a).

2. *(10 pts.)* Explain why $\forall x[x \text{ glitters} \;\Rightarrow\; \neg(x \text{ is gold})]$ is *not* a correct translation of "All that glitters is not gold." Provide a correct translation.

The intended meaning of the English sentence is, "It is not true that everything that glitters is gold." That is,

$$\neg\forall x[x \text{ glitters} \;\Rightarrow\; x \text{ is gold}].$$

By standard logic rules this formula can be converted to

$$\exists x \neg[\neg(x \text{ glitters}) \vee x \text{ is gold}]$$

and hence to

$$\exists x[x \text{ glitters} \wedge \neg(x \text{ is gold})]$$

(which many students wrote down immediately by common-sense reasoning). Any of these three is correct.

3. *(20 pts.)* Recall the three incidence axioms (two expressed in logical symbols to save space):
   1. $\forall \mathrm{P} \forall \mathrm{Q} \, [(\mathrm{P} \neq \mathrm{Q}) \ \Rightarrow \ \exists! l \, (\mathrm{P} \ \mathtt{I} \ l \ \wedge \ \mathrm{Q} \ \mathtt{I} \ l)]$
   2. $\forall l \exists \mathrm{P} \exists \mathrm{Q} \, [\mathrm{P} \neq \mathrm{Q} \wedge (\mathrm{P} \ \mathtt{I} \ l \ \wedge \ \mathrm{Q} \ \mathtt{I} \ l)]$
   3. There exist 3 distinct points that are not collinear.

   Consider the geometry consisting of 6 points, where the lines are the two-element subsets.

   (a) Verify that the incidence axioms are satisfied. (Discuss each axiom in English, not logical symbolism.)

Since all lines are 2-element sets, I-2 is obviously true (every line contains (at least) 2 points). I-1 is true because $\{\mathrm{P}, \mathrm{Q}\}$ contains P and Q and is the *only* 2-element set containing both of them. Finally, since each line has only 2 points, and there are more than 2 points in the space, there must be points that are not on the same line.

   (b) What parallelism property (if any) does this model satisfy?

hyperbolic. (There are 4 lines parallel to any given line.)

4–6. *(each 20 pts.)* Do **THREE** of these: *(no more than 10 points extra credit for doing all 4)*

   (A) State the four betweenness axioms.

See pp. 108 and 110–111.

   (B) Prove the ASA triangle congruence theorem, fully justifying all steps.

See pp. 127 and 151–152.

   (C) Define an *equivalence relation* and list 3 examples of equivalence relations closely related to the Hilbert axioms.

Am equivalence relation "$\simeq$" is a binary relation (i.e., relation between two things) that is (universal quantifiers implied)

(1) reflexive: $x \simeq x$
(2) symmetric: $x \simeq y \ \Rightarrow \ y \simeq x$
(3) transitive: $x \simeq y \wedge y \simeq z \ \Rightarrow \ x \simeq z$

The 3 examples I had in mind were

1. congruence of segments
2. congruence of angles
3. being on "the same side" of a line

Other possibilities include

4. congruence of triangles
5. equality

"Betweenness" does not qualify, because it is a relation among *three* things.

   (D) State and prove Pasch's Theorem.

See p. 114.

(Answer on separate paper; put your name on it and staple it to your test.)