Proof of Angle Ordering

Math 467 Writing Assignment 4

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To assist in the proof, we will first prove a Lemma which is an analog of proposition 3.12:

Lemma. Given $\angle ABC \cong \angle DEF$, then for any ray $\overrightarrow{BG}$ between $\overrightarrow{BA}$ and $\overrightarrow{BC}$, there is a unique ray $\overrightarrow{EH}$ between $\overrightarrow{ED}$ and $\overrightarrow{EF}$ such that $\angle GBC \cong \angle HEF$.

Proof. Assume that $AB \cong ED$ and $BC \cong EF$. By SAS, $\Delta ABC \cong \Delta DEF$. Then $AC \cong DF$ and $\angle BCA \cong \angle EFD$. By the crossbar theorem, $\overrightarrow{BG}$ intersects AC. Let G be the point of intersection. By proposition 3.12, there is a point H between D and F such that $CG \cong EH$. By SAS, $\Delta CBG \cong \Delta FEH$. Therefore, $\angle GBC \cong \angle HEF$. □

Proof of the angle ordering:

a. Exactly one of the following three conditions holds: $\angle ABC < \angle DEF$, $\angle ABC \cong \angle DEF$, or $\angle DEF < \angle ABC$

   Proof. Suppose $\angle ABC < \angle DEF$, then there is a ray, $\overrightarrow{EG}$ between $\overrightarrow{DE}$ and $\overrightarrow{EF}$ such that $\angle ABC \cong \angle GEF$.

   If $\angle ABC \cong \angle DEF$, then $\angle DEF \cong \angle GEF$, a contradiction.

   If $\angle ABC > \angle DEF$, then there is a ray $\overrightarrow{BH}$ between $\overrightarrow{BA}$ and $\overrightarrow{BC}$ such that $\angle HBC \cong \angle DEF$. Now, $\angle ABC \cong \angle GEF$, and $\overrightarrow{BH}$ is between $\overrightarrow{BA}$ and $\overrightarrow{BC}$. By the lemma, there is a ray $\overrightarrow{EI}$ between $\overrightarrow{EG}$ and $\overrightarrow{EF}$ such that $\angle HBC \cong \angle IEF$. But $\angle HBC \cong \angle DEF$, so we have $\angle IEF \cong \angle DEF$, $I \neq D$, a contradiction.

   Similarly, if one of the cases holds, then the other two cannot hold.

Suppose that none of the three conditions hold. Then there is no ray $\overrightarrow{EJ}$ emanating from E such that $\angle ABC \cong \angle JEF$, contradicting C-4. Therefore, exactly one of the conditions holds. $\square$

b. If $\angle ABC < \angle DEF, \angle DEF \cong \angle GHI$, then $\angle ABC < \angle GHI$.

Proof. Let $\angle ABC < \angle DEF$ and $\angle DEF \cong \angle GHI$. Then there is a ray $\overrightarrow{EJ}$ between $\overrightarrow{ED}$ and $\overrightarrow{EF}$ such that $\angle ABC \cong \angle JEF$. By the lemma, there is a ray $\overrightarrow{HK}$ between $\overrightarrow{HI}$ and $\overrightarrow{HG}$ such that $\angle JEF \cong \angle KHI$. But $\angle JEF \cong ABC$, so $\angle KHI \cong ABC$. Therefore, $\angle ABC < \angle GHI$. $\square$

c. If $\angle ABC > \angle DEF, \angle DEF \cong \angle GHI$, then $\angle ABC > \angle GHI$.

Proof. Let $\angle ABC > \angle DEF$ and $\angle DEF \cong \angle GHI$. Then there is a ray $\overrightarrow{BJ}$ between $\overrightarrow{BA}$ and $\overrightarrow{BC}$ such that $\angle JBC \cong \angle DEF$. But $\angle DEF \cong \angle GHI$, so $\angle JBC \cong \angle DEF$. Therefore, $\angle ABC > \angle GHI$. $\square$

d. If $\angle ABC < \angle DEF, \angle DEF < \angle GHI$, then $\angle ABC < \angle GHI$.

Proof. Let J be a point such that $\angle JEF \cong \angle ABC$ and K be a point such that $\angle KHI \cong \angle DEF$. Let L be a point such that $\overrightarrow{BA}$ is between $\overrightarrow{BL}$ and $\overrightarrow{BC}$ such that $\angle LBA \cong \angle DEJ$. Then $\angle LBC \cong \angle DEF \cong \angle KHI$ by angle addition. Let M be a point such that $\overrightarrow{BL}$ is between $\overrightarrow{BM}$ and $\overrightarrow{BC}$ such that $\angle MBL \cong \angle GHK$. Then $\angle MBC \cong \angle GHI$ by angle addition. Now $\angle ABC < \angle MBC$ and $\angle MBC \cong \angle GHI$, so $\angle ABC < \angle GHI$ by part (b). $\square$