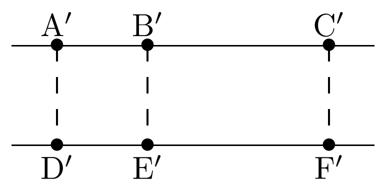
Proof of Proposition 3.13 (Exercises 21–23)

First, to make the logical substitutions less confusing, let's rewrite Prop. 3.12 and the definitions of $<$ and $>$ with primed letters:

Proposition 3.12: Given $A'C' \cong D'F'$, for any point B' between A' and C' there is a unique point E' between D' and F' such that $A'B' \cong D'E'$.



Definition of segment ordering: $A'B' < C'D'$ (or $C'D' > A'B'$) means that there is a point E' such that $C' * E' * D'$ and $A'B' \cong C'E'$.

(a) Prove that exactly one of $AB < CD$, $AB \cong CD$, $AB > CD$ holds.

By Axiom C-1 there is a unique F on $\overrightarrow{CD}$ with $AB \cong CF$. By definition of a ray, there are three cases (mutually exclusive):

1. $F = D$. Then $AB \cong CD$ (and conversely, by the uniqueness of F).
2. $C * F * D$. Then $AB < CD$ (and conversely) by definition of $<$ (with $E' = F$, etc.).
3. $C * D * F$. In Prop. 3.12 let $A' = C$, $B' = D$, $C' = F$, $D' = A$, $F' = B$:

$$CF \cong AB \wedge C * D * F \;\Rightarrow\; \exists!\, E' \colon A * E' * B \wedge CD \cong AE'.$$

Because the two hypotheses hold, the two conclusions hold; they say that $AB > CD$ (by the definition with $A' = C$, $B' = D$, $C' = A$, $D' = B$). [If you don't like what I'm about to say, look at the alternative proof after the proof of (d).] Conversely, if $AB > CD$, then such an E' exists; we have $CD \cong AE'$ and $A * E' * B$ as well as $CF \cong AB$. If $F = D$, then we have shown that $AB \cong CD$, which contradicts the uniqueness of E' (as guaranteed by Axiom C-1). If $C * F * D$, then because $CD \cong AE'$ there is a G between A and E' with $AG \cong CF$ (by 3.12 again). But then $AG \cong AB$ by transitivity, hence $G = B$ by the uniqueness part of C-1. Thus $A * B * E'$ and $A * E' * B$, contradicting Axiom B-3. (*Remark:* Here we have proved a sort of converse to Axiom C-3.) So $C * D * F$ is the only possibility.

(b) Prove that $AB < CD \wedge CD \cong EF \;\Rightarrow\; AB < EF$.

There is a G with $C * G * D$ and $AB \cong CG$. Apply Prop. 3.12 with $A' = C$, $B' = G$, $C' = D$, $D' = E$, $E' = H$, $F' = F$:

$$\exists H \colon E * H * F \wedge CG \cong EH.$$

Hence $AB \cong EH$, which implies $AB < EF$ by the definition.

(c) Prove that $AB > CD \wedge CD \cong EF \Rightarrow AB > EF$.

There is a point H such that A * H * B and AH ≅ CD. Then AH ≅ EF by transitivity of ≅ (Axiom C-2). So by definition of <, we have EF < AB.

(d) Prove that $AB < CD \wedge CD < EF \Rightarrow AB < EF$.

By hypothesis, there is a G such that $C * G * D \wedge AB \cong CG$, and there is an H such that $E * H * F \wedge CD \cong EH$. By Prop. 3.12, there is a point I such that $E * I * H \wedge CG \cong EI$. By Prop. 3.3 and transitivity of congruence, therefore, $E * I * F \wedge AB \cong EI$, which says precisely that AB < EF.

Remark: The proof of (d) does not use (a), so we may use (d) to provide an alternative to the awkward "Conversely ... " part of the proof of (a.3): If AB > CD (i.e., CD < AB) and also AB < CD, then by (d), AB < AB, which is false. (By the uniqueness statement in Axiom C-1 and the distinctness statement in Axiom B-1, we can't have both AB ≅ AB and AB ≅ AE with $A * E * B$.) If AB > CD and also AB ≅ CD, then we get essentially the same contradiction. This completes the proof that only one of the three conditions can hold.