Fall 2002 Math 151 Common Exam 1A Solutions Wed, 09/Oct/2002

1-a We have $2\mathbf{a} - 3\mathbf{b} = 2\langle 2, 3\rangle - 3\langle 1, -2\rangle = \langle 4, 6\rangle + \langle -3, 6\rangle = \langle 1, 12\rangle$.

2-b The two vectors are parallel if and only if one is a multiple of the other. Hence $\langle 2, x\rangle = c\langle -3, 4\rangle$ from which $-3c = 2$ and $4c = x$. These equations yield $c = -\frac{2}{3}$ and $x = -\frac{8}{3}$.

3-b From the diagram, we have $\mathbf{b} + \mathbf{c} = \mathbf{a}$, whence (ii) $\mathbf{a} - \mathbf{b} = \mathbf{c}$. This is the only true statement among the four.

4-c With $\mathbf{r}(t) = \langle x(t), y(t)\rangle = \langle \cos t, 2\sin t\rangle$, we have $1 = \cos^2 t + \sin^2 t = x^2 + (y/2)^2$, whence $4x^2 + y^2 = 4$.

5-c Recall that if w is negative, then $|w| = -w$. Accordingly, $\displaystyle\lim_{x\to 1^+} \frac{x-1}{|1-x|} = \lim_{x\to 1^+} \frac{x-1}{-(1-x)} = \lim_{x\to 1^+} 1 = 1$.

6-b Similarly, if w is positive (or zero), then $|w| = w$. Moreover, $\sqrt{u^2} = |u|$ for any real number u. Thus

$$\lim_{x\to\infty} \frac{\sqrt{4x^2+1}+1}{x} = \lim_{x\to\infty} \frac{|x|\sqrt{4+\frac{1}{x^2}}+1}{x} = \lim_{x\to\infty} \frac{\sqrt{4+\frac{1}{x^2}}+\frac{1}{x}}{1} = 2$$

7-d Clearly $\displaystyle\lim_{x\to 1} f(x) = 2$ is false since the left-hand limit $\displaystyle\lim_{x\to 1^-} f(x) = 1$ and right-hand limit $\displaystyle\lim_{x\to 1^+} f(x) = 2$ differ.

8-e The statement *f is continuous at c* is the same as saying $\displaystyle\lim_{x\to c} f(x) = f(c)$.

9-c Equivalently, which of the stated intervals must contain a solution of the equation $f(x) = x^5 + 2x^3 - 1 = 0$? First note that $f(0) = -1 < 0$ and $f(1) = 2 > 0$. Next, recall that $f(x)$, a polynomial, is continuous everywhere. Applying the Intermediate Value Theorem (IVT), we conclude that there is a c in $[0, 1]$ such that $f(c) = 0$ and thus $c^5 = 1 - 2c^3$.

10-c Rewrite f as $f(x) = 1 + x^{3/2}$. Then $f'(x) = \frac{3}{2}x^{1/2}$, whence $f'(1) = \frac{3}{2}$.

11-d Rewrite the given line as $y = \frac{2}{3}x + \frac{4}{3}$, the slope of which is $m = \frac{2}{3}$. If the tangent line to $y = f(x)$ at $x = 1$ is perpendicular to the given line, then its slope is $f'(1) = -1/m = -\frac{3}{2}$.

12-c As $x \to 2$, we have $\dfrac{(f(x))^2 - 9}{x-2} = \dfrac{(f(x)-3)}{x-2}(f(x)+3) \to f'(2)\cdot(f(2)+3) = 3\cdot(3+3) = 18$.

Alternatively, let $g = f^2 = f\cdot f$. Then $\displaystyle\lim_{x\to 2} \frac{(f(x))^2-9}{x-2} = \lim_{x\to 2} \frac{(f(x))^2-(f(2))^2}{x-2} = \lim_{x\to 2} \frac{g(x)-g(2)}{x-2} = g'(2)$. Now use the product rule to obtain $g'(2) = \big(f(x)f'(x) + f'(x)f(x)\big)\big|_{x=2} = (3)(3) + (3)(3) = 18$.

13. The triangle has vertices $A(1, 1)$, $B(3, 4)$, and $C(2, -1)$. Thus $\overrightarrow{AB} = \vec{B} - \vec{A} = \langle 2, 3\rangle$. Similarly, we have $\overrightarrow{AC} = \langle 1, -2\rangle$ and $\overrightarrow{BC} = \langle -1, -5\rangle$. Moreover, $\overrightarrow{BA} = -\overrightarrow{AB} = \langle -2, -3\rangle$.

(a) The (physics) definition of dot product $\overrightarrow{BA}\cdot\overrightarrow{BC} = \left\|\overrightarrow{BA}\right\|\left\|\overrightarrow{BC}\right\|\cos\theta$ yields

$$\cos\theta = \frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{\left\|\overrightarrow{BA}\right\|\left\|\overrightarrow{BC}\right\|} = \frac{2+15}{\sqrt{13}\sqrt{26}} = \frac{17}{13\sqrt{2}} \text{ or } \frac{17\sqrt{2}}{26}\ (\approx 0.92)$$

(b) Triangle ADB is a right triangle. Accordingly, the length of side BD is $\left\|\overrightarrow{BA}\right\|\cos\theta$. Therefore, by the Pythagorean Theorem, the length of AD is $\sqrt{\left\|\overrightarrow{BA}\right\|^2 - \left(\left\|\overrightarrow{BA}\right\|\cos\theta\right)^2} = \left\|\overrightarrow{BA}\right\|\sqrt{1-\cos^2\theta} = \sqrt{13}\sqrt{1-\frac{17^2}{13^2\cdot 2}} = \sqrt{13-\frac{17^2}{26}} = \sqrt{\frac{49}{26}} = \frac{7}{\sqrt{26}}$ or $\frac{7\sqrt{26}}{26}$ (≈ 1.37).

Alternatively, we may use projections: $\left\|\overrightarrow{DA}\right\| = \left\|\text{orth}_{\overrightarrow{BC}}\overrightarrow{BA}\right\| = \left\|\overrightarrow{BA} - \text{proj}_{\overrightarrow{BC}}\overrightarrow{BA}\right\| = \left\|\overrightarrow{BA} - \left(\dfrac{\overrightarrow{BC}\cdot\overrightarrow{BA}}{\left\|\overrightarrow{BC}\right\|}\right)\dfrac{\overrightarrow{BC}}{\left\|\overrightarrow{BC}\right\|}\right\|$ $= \left\|\langle -2, -3\rangle - \frac{17}{\sqrt{26}}\frac{\langle -1, -5\rangle}{\sqrt{26}}\right\| = \left\|\langle -2, -3\rangle - \left\langle -\frac{17}{26}, -\frac{85}{26}\right\rangle\right\| = \left\|\left\langle -\frac{35}{26}, \frac{7}{26}\right\rangle\right\| = \frac{7\sqrt{26}}{26}$ (≈ 1.37).

(c) An equation of the line passing through A and C is $\vec{\gamma}(t) = \vec{A} + t\,\overrightarrow{AC} = \langle 1, 1\rangle + t\,\langle 1, -2\rangle = \langle 1+t, 1-2t\rangle$.

14. With c a fixed real number, let $f(x) = \begin{cases} cx+2 & \text{if } x < 2 \\ 0 & \text{if } x = 2 \\ c^2+2cx+1 & \text{if } x > 2 \end{cases}$.

(a) The left-hand limit of f at $x = 2$ is $\lim_{x\to 2^-} f(x) = 2c+2$.

(b) The right-hand limit of f at $x = 2$ is $\lim_{x\to 2^+} f(x) = c^2+4c+1$.

(c) For $\lim_{x\to 2} f(x)$ to exist, the one-sided limits in (a) and (b) must match up: $2c+2 = c^2+4c+1$ or $c^2+2c-1 = 0$. Via the quadratic formula, we have $c = \dfrac{-2\pm\sqrt{4+4}}{2} = -1\pm\sqrt{2}$ ($\approx 0.41, -2.41$).

(d) For the values of c obtained in (c), use either one-sided limit to obtain the two-sided limit. Clearly the left-hand limit is the easier to apply: $(2c+2)|_{c=-1\pm\sqrt{2}} = \pm 2\sqrt{2}$ ($\approx \pm 2.83$).

(e) Assume that there is a value of c for which f is continuous at $x = 2$. Then $0 = f(2) = \lim_{x\to 2} f(x) = \pm 2\sqrt{2}$; i.e., $0 = \pm 2\sqrt{2}$, a contradiction. Accordingly, there is *no* value of c for which f is continuous at $x = 2$.

15. Let $f(x) = \sqrt{5+2x}$.

(a) In order for $f(x)$ to be defined, we require the expression under the square root to be nonnegative; i.e., $5+2x \geq 0$, whence $2x \geq -5$ or $x \geq -\frac{5}{2}$. Therefore, the domain of f is $D = \left\{x \in \mathbb{R} : x \geq -\frac{5}{2}\right\}$.

(b) The derivative definition gives $f'(-1) = \lim_{x\to -1} \dfrac{f(x)-f(-1)}{x-(-1)} = \lim_{x\to -1} \dfrac{\sqrt{5+2x}-\sqrt{3}}{x+1} = \lim_{x\to -1} \dfrac{(5+2x)-3}{(x+1)\left(\sqrt{5+2x}+\sqrt{3}\right)}$

$= \lim_{x\to -1} \dfrac{2\,(x+1)}{(x+1)\left(\sqrt{5+2x}+\sqrt{3}\right)} = \lim_{x\to -1} \dfrac{2}{\sqrt{5+2x}+\sqrt{3}} = \dfrac{2}{2\sqrt{3}} = \dfrac{1}{\sqrt{3}}$ or $\dfrac{\sqrt{3}}{3}$ (≈ 0.58).

Alternatively, $f'(-1) = \lim_{h\to 0} \dfrac{f(-1+h)-f(-1)}{h} = \lim_{h\to 0} \dfrac{\sqrt{5+2(-1+h)}-\sqrt{3}}{h} = \lim_{h\to 0} \dfrac{(5+2(-1+h))-3}{h(\sqrt{5+2(-1+h)}+\sqrt{3})}$

$= \lim_{h\to 0} \dfrac{2h}{h(\sqrt{5+2(-1+h)}+\sqrt{3})} = \lim_{h\to 0} \dfrac{2}{\sqrt{5+2(-1+h)}+\sqrt{3}} = \dfrac{2}{2\sqrt{3}} = \dfrac{1}{\sqrt{3}}$ or $\dfrac{\sqrt{3}}{3}$ (≈ 0.58).

16. The particle's position is $s = \dfrac{\sqrt{t}}{t^2+1} = \dfrac{t^{1/2}}{t^2+1}$.

(a) For $t > 0$, its velocity is $v = \dfrac{ds}{dt} = \dfrac{(t^2+1)\left(\frac{1}{2}t^{-1/2}\right) - t^{1/2}(2t)}{(t^2+1)^2}$. You may simplify this as $\dfrac{t^2+1-4t^2}{2\sqrt{t}\,(t^2+1)^2} = \dfrac{1-3t^2}{2\sqrt{t}\,(t^2+1)^2}$.

(b) For $t > 0$, the velocity is zero when $\dfrac{1-3t^2}{2\sqrt{t}\,(t^2+1)^2} = 0$. This occurs when $1-3t^2 = 0$ or $t = \pm\frac{1}{\sqrt{3}} = \pm\frac{\sqrt{3}}{3}$. Recall, however, that $t > 0$. Hence $t = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ (≈ 0.58) is the only time at which the velocity is zero.

17. Let f and g be differentiable functions whose values and derivatives at $x = 1$ are as stated in the table on your exam. Define $U(x) = \dfrac{1}{f(x)}$, $V(x) = g(x)\,f(x)$, and $S(x) = f^2(x) = (f(x))^2 = f(x)\,f(x)$.

(a) Then $U'(x) = \dfrac{f(x)\cdot 0 - 1\cdot f'(x)}{(f(x))^2} \overset{@x=1}{=} \dfrac{2\cdot 0 - 1\cdot 3}{2^2} = -\dfrac{3}{4}$.

(b) Moreover, $V'(x) = g(x)\,f'(x) + g'(x)\,f(x) \overset{@x=1}{=} (-1)\,(3) + (2)\,(2) = 1$.

(c) Finally, $S'(x) = f(x)\,f'(x) + f'(x)\,f(x) \overset{@x=1}{=} (2)\,(3) + (3)\,(2) = 12$.