Fall 2002 Math 151 Common Exam 2A Solutions Wed, 06/Nov/2002

1-c We have $y' = 2 + \sec x \tan x = 2$ at $x = 0$.

2-d Via the power rule and chain rules, we have $f'(x) = 2\cos x(-\sin x) = -2\cos x \sin x$.

3-c We have $f'(x) = -e^{-x} = -1$ at $x = 0$.

4-b By continuity, direct substitution yields $\lim_{\theta\to 0} \frac{\sin\theta}{\pi-\theta} = \frac{0}{\pi-0} = 0$.

5-b As $x \to \infty$, we have $\frac{1}{x} \to 0$, whence $\lim_{x\to\infty} \cos\left(\frac{1}{x}\right) = \cos 0 = 1$.

6-a Regarding $y = y(x)$ as a function of x, differentiate $y^3 - x^2 = 1 - 2y$ with respect to x. This yields $3y^2\frac{dy}{dx} - 2x = -2\frac{dy}{dx}$ or $(3y^2+2)\frac{dy}{dx} = 2x$. Therefore, $\frac{dy}{dx} = \frac{2x}{3y^2+2}$.

7-b We have velocity $v = s' = 6t^2 - 18t$ and acceleration $a = v' = s'' = 12t - 18$. Hence acceleration $12t - 18 = 0$ when $t = 3/2$.

8-c With $x(t) = t^3$ and $y(t) = \sin(\pi t)$, we have $t = 1$ when $(x, y) = (1, 0)$. Accordingly, the slope of the tangent line to the curve when $t = 1$ is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\pi\cos(\pi t)}{3t^2} = \frac{\pi\cos(\pi)}{3} = \frac{\pi\cdot(-1)}{3} = -\frac{\pi}{3}$$

9-e Rewrite f as $f(x) = \left(1+x^2\right)^{1/2}$. Then $f'(x) = \frac{1}{2}\left(1+x^2\right)^{-1/2}\cdot 2x$, whence $f'(0) = 0$. Therefore, the tangent-line (linear) approximation to $f(x)$ at $x = 0$ is $y = f(0) + f'(0)(x-0) = 1 + 0\cdot(x-0) = 1$ or $y = 1$.

10-d The function $f(x) = x^2$, $D = [-1, 1]$ is **not** one-to-one since, for example, $f(-1) = f(1) = 1$.

11-a At time t, let y be the distance from the ant to the origin $O(0, 0)$ and z be the distance from the ant to the point $P(1, 0)$. By the Pythagorean Theorem, we have $z^2 = y^2 + 1^2$. Differentiating with respect to t gives $2z\frac{dz}{dt} = 2y\frac{dy}{dt}$, whence for $y = 3$:

$$\frac{dz}{dt} = \frac{y}{z}\frac{dy}{dt} = \frac{3}{\sqrt{3^2+1^2}} \times \frac{1}{2} = \frac{3}{2\sqrt{10}} = \frac{3\sqrt{10}}{20}$$

12-e Given $H(x) = f(x/2)$, repeated application of the chain rule gives $H'(x) = \frac{1}{2}f'(x/2)$, $H''(x) = \frac{1}{2^2}f''(x/2)$, and in general $H^{(n)}(x) = \frac{1}{2^n}f^{(n)}(x/2)$. Therefore, $H^{(15)}(0) = \frac{1}{2^{15}}f^{(15)}(0) = \frac{2}{2^{15}} = 2^{-14}$.

13. Regarding $y = y(x)$ as a function of x, differentiate $x^3 - x^2y^2 + 3y - 1 = 0$ with respect to x.

(i) This yields $3x^2 - \left(x^2\cdot 2y\frac{dy}{dx} + y^2\cdot 2x\right) + 3\frac{dy}{dx} = 0$, whence $(3-2x^2y)\frac{dy}{dx} = 2xy^2 - 3x^2$. Therefore, $\frac{dy}{dx} = \frac{2xy^2-3x^2}{3-2x^2y}$.

(ii) At $(x, y) = (-1, 1)$, the slope of the tangent line is $m = \frac{dy}{dx} = \frac{-2-3}{3-2} = -5$. Via the point-slope formula, an equation of the tangent line is $y - 1 = -5(x+1)$ or $y = -5x - 4$.

14. Rewrite $\mathbf{r}$ as $\mathbf{r}(t) = \left\langle e^{t^2+t}, (1-3t)^{1/2} \right\rangle$.

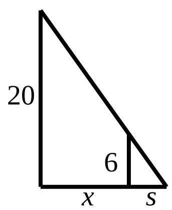
   (i) Then $\mathbf{r}'(t) = \left\langle e^{t^2+t}(2t+1), \frac{1}{2}(1-3t)^{-1/2}(-3) \right\rangle$, which you may write as $\left\langle (2t+1)e^{t^2+t}, -\dfrac{3}{2\sqrt{1-3t}} \right\rangle$.

   (ii) Hence when $t=0$, a tangent vector is $\mathbf{v} = \mathbf{r}'(0) = \left\langle 1, -\frac{3}{2} \right\rangle$.

   (iii) A point on the tangent line to the curve C corresponding to $t=0$ is $\vec{P} = \mathbf{r}(0) = \langle 1, 1 \rangle$. Accordingly, a vector equation of this tangent line is
   $$\vec{L}(u) = \vec{P} + u\mathbf{v} = \langle 1, 1 \rangle + u\left\langle 1, -\tfrac{3}{2} \right\rangle \text{ or } \vec{L}(u) = \left\langle u+1, 1-\tfrac{3}{2}u \right\rangle$$

15. A 6-foot tall woman walks at 5 ft/s in a straight path away from a street light atop a 20-foot tall pole.

   (i) At time t, let x be the distance from the pole to the woman and s the length of the woman's shadow.



   (ii) Via similar triangles, we have that $\dfrac{\text{height}}{\text{base}} = \dfrac{20}{s+x} = \dfrac{6}{s}$, whence $20s = 6s + 6x$. Thus $14s = 6x$ or $s = \frac{3}{7}x$. Accordingly, $\dfrac{ds}{dt} = \dfrac{3}{7}\dfrac{dx}{dt} = \dfrac{3}{7} \times 5 = \dfrac{15}{7}$ ft/s (approximately 2.14 ft/s) is the rate at which her shadow lengthens.

16. Let f be a differentiable function.

   (i) With $S(x) = 2e^{f(x)} - f(e^x)$, the chain rule gives $S'(x) = 2e^{f(x)}f'(x) - f'(e^x)e^x$.

   (ii) With $U(x) = f(x \sin x)$, the chain and product rules give $U'(x) = f'(x \sin x) \cdot (x \cos x + \sin x)$.

17. Let F and G be differentiable functions whose tangent-line (linear) approximations at $x = 2$ are $y = L_1(x) = 1 + 2x$ and $y = L_2(x) = 2 - 3x$, respectively.

   (i) The tangent-line approximation to a curve at a point has the same function value and derivative value as the curve at said point. Therefore
   $$\begin{aligned} F(2) &= L_1(2) = 5 \\ F'(2) &= L_1'(2) = 2 \\ G(2) &= L_2(2) = -4 \\ G'(2) &= L_2'(2) = -3 \end{aligned}$$

   (ii) Let $H = \dfrac{F}{G}$. Then $H' = \dfrac{GF' - FG'}{G^2}$. Hence $H'(2) = \dfrac{G(2)F'(2) - F(2)G'(2)}{(G(2))^2} = \dfrac{(-4)(2) - (5)(-3)}{(-4)^2} = \dfrac{7}{16}$. We also have $H(2) = \dfrac{F(2)}{G(2)} = \dfrac{5}{-4} = -\dfrac{5}{4}$. Accordingly, the tangent-line approximation to $H = \dfrac{F}{G}$ at $x = 2$ is
   $$y = L(x) = H(2) + H'(2)(x-2) = -\frac{5}{4} + \frac{7}{16}(x-2)$$
   That is, $y = -\frac{5}{4} + \frac{7}{16}(x-2)$ or $y = \frac{7}{16}x - \frac{17}{8}$.