Fall 2005 Math 151
Exam 3B: Solutions
Mon, 05/Dec ©2005, Art Belmonte

1. (e) Recall that velocity is the derivative of position and acceleration the derivative of velocity. Antidifferentiate acceleration to get velocity, then antidifferentiate velocity to get position, resolving constants along the way.

$$\begin{aligned}
\frac{d\mathbf{v}(t)}{dt} = \mathbf{a}(t) &= [1, 2t] \\
\mathbf{v}(t) &= \left[t, t^2\right] + \mathbf{C} \\
[0,1] = \mathbf{v}(0) &= [0,0] + \mathbf{C} \\
\mathbf{C} &= [0,1] \\
\frac{d\mathbf{r}(t)}{dt} = \mathbf{v}(t) &= \left[t, t^2+1\right] \\
\mathbf{r}(t) &= \left[\tfrac{1}{2}t^2, \tfrac{1}{3}t^3 + t\right] + \mathbf{K} \\
[-1,1] = \mathbf{r}(0) &= [0,0] + \mathbf{K} \\
\mathbf{K} &= [-1,1] \\
\mathbf{r}(t) &= \left[\tfrac{1}{2}t^2 - 1, \tfrac{1}{3}t^3 + t + 1\right] \\
\mathbf{r}(1) &= \left[-\tfrac{1}{2}, \tfrac{7}{3}\right]
\end{aligned}$$

2. (d) Given $g(x) = \sin^{-1} x$, we have

$$\begin{aligned}
g'(x) &= \frac{1}{\sqrt{1-x^2}} = \left(1-x^2\right)^{-1/2} \\
g''(x) &= -\tfrac{1}{2}\left(1-x^2\right)^{-3/2}(-2x) = \frac{x}{\left(1-x^2\right)^{3/2}} \\
g''\left(\tfrac{1}{2}\right) &= \frac{\frac{1}{2}}{\left(\frac{3}{4}\right)^{3/2}} = \frac{\frac{1}{2}}{\left(\frac{\sqrt{3}}{2}\right)^3} = \frac{4}{3\sqrt{3}} = \frac{4\sqrt{3}}{9}
\end{aligned}$$

3. (a) The Extreme Value Theorem (EVT) guarantees that the *continuous* polynomial $f(x) = x^3 - 12x + 5$ does indeed have absolute maximum and absolute minimum values on the *closed* interval $[-5, 3]$.

- Find the critical values of f. These are values of x for which f' is either zero or undefined. The derivative of f, another polynomial, exists for all real numbers. So we are looking for values of x for which $f'(x) = 0$.

$$\begin{aligned}
f'(x) = 3x^2 - 12 &= 0 \\
3\left(x^2 - 4\right) &= 0 \\
3(x-2)(x+2) &= 0 \\
x &= -2, 2
\end{aligned}$$

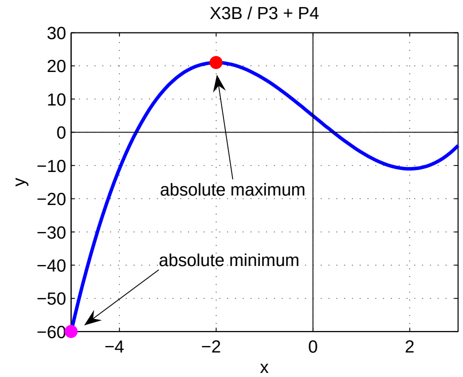
  Hence the critical numbers of f are $x = -2, 2$. Both of these x-values lie in the interval $[-5, 3]$.

- Now crank out function values of f at the critical values along with the endpoints of the closed interval.

$$\begin{aligned}
f(-5) &= -125 + 60 + 5 = -60 \\
f(-2) &= -8 + 24 + 5 = 21 \\
f(2) &= 8 - 24 + 5 = -11 \\
f(3) &= 27 - 36 + 5 = -4
\end{aligned}$$

  The absolute maximum is $f(-2) = 21$.

- Here is a graph that corroborates this assertion.

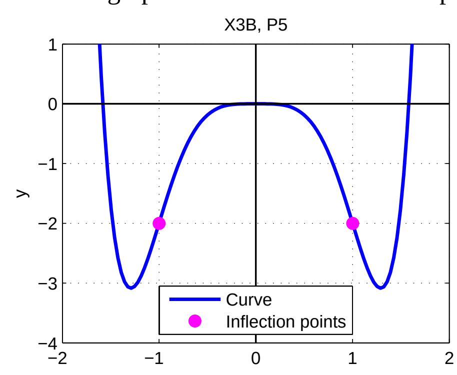


4. (b) From the analysis in Problem 3, we see that the absolute minimum of f on $[-5, 3]$ is $f(-5) = -60$.

5. (a) The inflection points of $g(x) = \frac{2}{3}\left(2x^6 - 5x^4\right)$ are values of x at which g'' changes sign. This may occur where g'' is either zero or undefined. Since g is a polynomial, g'' exists for all real numbers. So let's find where $g'' = 0$ to obtain our *candidates*.

$$\begin{aligned}
g'(x) &= \tfrac{2}{3}\left(12x^5 - 20x^3\right) \\
g''(x) &= \tfrac{2}{3}\left(60x^4 - 60x^2\right) = 40x^2\left(x^2 - 1\right) = 0 \\
40x^2(x-1)(x+1) &= 0 \\
x &= 0, \pm 1
\end{aligned}$$

- As x increases through -1, g'' changes from $+$ to $-$.
- As x increases through 1, g'' changes from $-$ to $+$.
- However, as x increases through 0, g'' stays negative. That is, it does NOT change sign.
- Accordingly, the *two* inflection points of g are $x = \pm 1$.
- Here is a graph that shows the inflection points.



6. (d) Use L'Hospital's Rule and some algebra.

$$\begin{aligned}
\lim_{x\to 0} \frac{\ln(1-x) + x + \frac{1}{2}x^2}{x^3} &\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{\frac{-1}{1-x} + 1 + x}{3x^2} \\
&= \lim_{x\to 0} \frac{-1 + 1 - x^2}{3x^2(1-x)} \\
&= \lim_{x\to 0} \frac{-1}{3(1-x)} = -\frac{1}{3}
\end{aligned}$$

[Alternatively, you may use L'Hospital's Rule *three* times.]

7. (d) In English, $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is "the angle (between $\pm\frac{\pi}{2}$) whose sine is $\frac{\sqrt{3}}{2}$." This angle is $\frac{\pi}{3}$ radians or 60°.

8. (c) Let $y = x^x$. Use logarithmic differentiation.

$$\begin{aligned} \ln y &= x \ln x \\ \frac{1}{y}\frac{dy}{dx} &= (1)\ln x + (x)\frac{1}{x} \\ \frac{dy}{dx} &= x^x(1 + \ln x) \end{aligned}$$

9. (c) The area under the curve $y = f(x) = 1 + x^3$ above the x-axis between $x = a = 2$ and $x = b = 6$ may be formulated as the limit of a Riemann sum that uses subintervals of equal length and function values at right endpoints. Let's construct this limit of a sum one step at a time.

$$\begin{aligned} \Delta x &= \frac{b-a}{n} = \frac{6-2}{n} = \frac{4}{n} \\ x_i &= a + i\,\Delta x = 2 + \frac{4i}{n} \\ f(x_i) &= 1 + \left(2 + \frac{4i}{n}\right)^3 \\ \text{area} &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x \\ \text{area} &= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}\left(1 + \left(2 + \frac{4i}{n}\right)^3\right) \end{aligned}$$

10. (a) The Mean Value Theorem (MVT) guarantees that there is a number $c \in (1, 3)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{f(3) - f(1)}{3 - 1} = \frac{8 - 5}{2} = \frac{3}{2}.$$

11. (e) Use the law of exponential growth.

$$\begin{aligned} y &= y_0 e^{kt} \\ 9000 &= 1000e^{2k} \\ 9 &= e^{2k} \\ \ln 9 &= 2k \\ k &= \tfrac{1}{2}\ln 9 = \ln\left(9^{1/2}\right) = \ln 3 \\ y &= 1000e^{t\ln 3} = 1000\left(e^{\ln 3}\right)^t = 1000(3)^t \end{aligned}$$

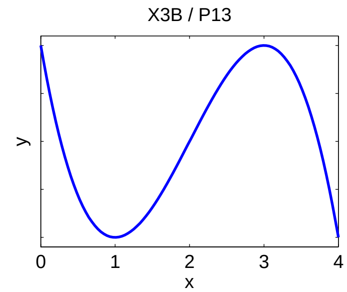
After 3 hours, the number of bacteria is $1000(3)^3 = 27{,}000$.

12. (b) G is an antiderivative of g provided $G'(x) = g(x)$. Since $\frac{d}{dx}\left(x^4 + \frac{1}{2}e^{x^2}\right) = 4x^3 + \frac{1}{2}e^{x^2}(2x) = 4x^3 + xe^{x^2}$, $G(x) = x^4 + \frac{1}{2}e^{x^2}$ is an antiderivative of the stated function $g(x) = 4x^3 + xe^{x^2}$.

13. (c) Consider the geometrical meaning conveyed by the information given about the first and second derivatives.

- $f'(x) > 0$ on $(1, 3)$ tells where f is increasing.
- $f'(x) < 0$ on $(0, 1) \cup (3, 4)$ tells where f is decreasing.
- $f''(x) > 0$ on $(0, 2)$ indicates where f is concave up.
- $f''(x) < 0$ on $(2, 4)$ shows where f is concave down.

These facts enable us to sketch a graph of f.



14. The limit $\lim_{x\to 0}(\cos 2x)^{1/x^2}$ involves an indeterminate power, 1^∞. We use the four-step procedure from Section 4.8.

- Let $y = (\cos 2x)^{1/x^2}$.
- Then $\ln y = (1/x^2)(\ln(\cos 2x)) = \dfrac{\ln(\cos 2x)}{x^2}$, an indeterminate quotient $0/0$ as $x \to 0$.
- Therefore,

$$\begin{aligned} \lim_{x\to 0}\frac{\ln(\cos 2x)}{x^2} &\overset{\text{L'H}}{=} \lim_{x\to 0}\frac{-2\sin 2x/\cos 2x}{2x} = \lim_{x\to 0}\frac{-2\tan 2x}{2x} \\ &\overset{\text{L'H}}{=} \lim_{x\to 0}\frac{-4\sec^2 2x}{2} \\ \lim_{x\to 0}\ln y &= -2. \end{aligned}$$

- Hence $\lim_{x\to 0}(\cos 2x)^{1/x^2} = \lim_{x\to 0} y = \lim_{x\to 0} e^{\ln y} = e^{-2}$.

15. Let $f(x) = x^3 - 3x^2 + 3$.

(a) The critical points of f are where f' is zero or undefined. Since f is a polynomial, its derivative is defined for all values of x. So let's determine where f' is zero. We have

$$f'(x) = 3x^2 - 6x = 3x(x - 2) = 0$$

whence $x = 0, 2$.

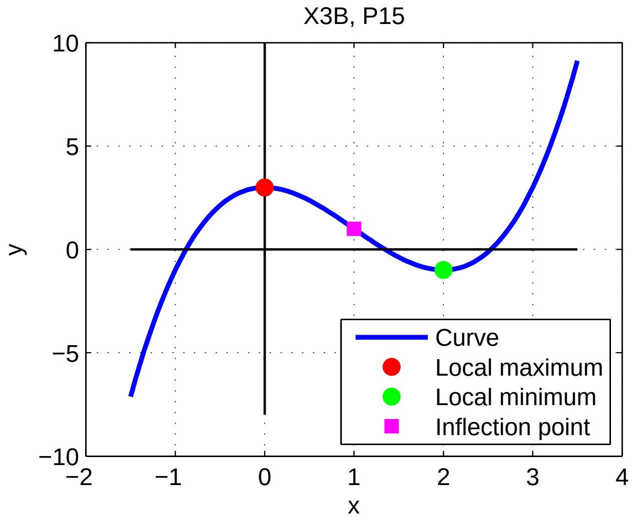
(b) Use the First Derivative Test to classify each critical point as being the location of a local maximum, local minimum, or neither.

- As x increases through 0, the sign of f' changes from $+$ to $-$. This signifies a local maximum at $x = 0$.
- As x increases through 2, the sign of f' changes from $-$ to $+$. This signifies a local minimum at $x = 2$.

(c) To find the inflection points of f, determine where the second derivative changes sign.

- Now $0 = f''(x) = 6x - 6$, whence $x = 1$.
- As x increases through 1, the sign of f'' changes from $-$ to $+$. Since the second derivative changes sign, $x = 1$ is an inflection point of f.

These assertions are corroborated by the following graph.



- NOTE: Instead of using the First Derivative Test in part (b), we may use the Second Derivative Test. Observe that $f''(x) = 6x - 6$ is continuous. Since $f'(0) = 0$ and $f''(0) = -6 < 0$, we conclude that f has a local maximum at $x = 0$. Similarly, since $f'(2) = 0$ and $f'(2) = 6 > 0$, we deduce that f has a local minimum at $x = 2$.

16. (a) Given $y = \tan^{-1} \sqrt{x^5 + 4}$, we have

$$\frac{dy}{dx} = \frac{1}{1 + \left(\sqrt{x^5+4}\right)^2} \cdot \frac{1}{2}\left(x^5+4\right)^{-1/2}\left(5x^4\right)$$

or $\dfrac{dy}{dx} = \dfrac{5x^4}{2\sqrt{x^5+4}\left(x^5+5\right)}$.

(b) Given $y = \ln\left(\ln\left(1+\sqrt{x}\right)\right)$, we have

$$\frac{dy}{dx} = \frac{1}{\ln\left(1+\sqrt{x}\right)} \cdot \frac{1}{1+\sqrt{x}} \cdot \frac{1}{2}x^{-1/2}$$

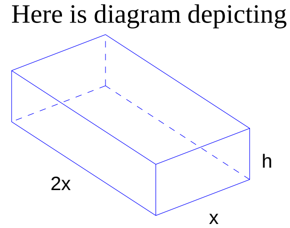
or $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}\left(1+\sqrt{x}\right)\ln\left(1+\sqrt{x}\right)}$.

(c) Use logarithmic differentiation.

$$\begin{aligned}
y &= \left(x+x^2\right)^{\tan x} \\
\ln y &= (\tan x)\ln\left(x+x^2\right) \\
\frac{1}{y}\frac{dy}{dx} &= \left(\sec^2 x\right)\ln\left(x+x^2\right) + (\tan x)\frac{1+2x}{x+x^2} \\
\frac{dy}{dx} &= \left(x+x^2\right)^{\tan x}\left(\left(\sec^2 x\right)\ln\left(x+x^2\right) + \frac{(1+2x)\tan x}{x+x^2}\right)
\end{aligned}$$

17. Let x be the width of the bottom of the box and h be the height of the box.

- Here is diagram depicting the situation.

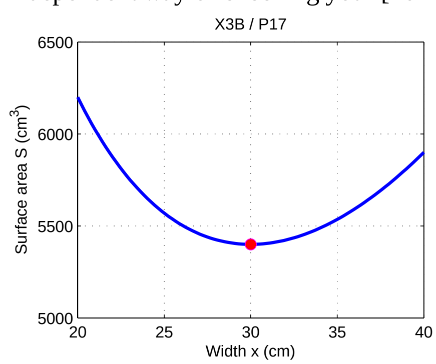


- The volume of the box is

$$V = (2x)(x)(h) = 2hx^2 = 36{,}000,$$

whence $h = 18{,}000x^{-2}$, $x > 0$.

- The surface area is $S = 2x^2 + 2hx + 2(2x)h$ or $S = 2x^2 + 6hx = 2x^2 + 108{,}000x^{-1}$, $x > 0$.
- Thus $S' = 4x - 108{,}000x^{-2} = 0$. So $x^3 = 27{,}000$ whence $x = 30$. Hence $x = 30$ cm is the width of the base that ostensibly minimizes the total surface area.
- Let's verify this assertion. The second derivative of S is $S'' = 4 + 216{,}000x^{-3} > 0$ for all $x > 0$. That is, S is concave up on its domain. Accordingly, the absolute minimum of occurs when $x = 30$ cm. [This minimum is $S(30) = 5{,}400$ cm^2.] The corresponding length is $2x = 60$ cm and the height is $h = \dfrac{18{,}000}{900} = 20$ cm.
- Here is a graph of S versus x. The problem is readily solved interactively using the graphing analysis capabilities of a TI-89. This gives an intuitive and independent way of checking your [home]work!



18. Velocity is the derivative of position and acceleration the derivative of velocity. Antidifferentiate acceleration to get velocity, then antidifferentiate velocity to get position, resolving constants along the way. Deceleration signifies negative acceleration.

$$\begin{aligned}
\frac{dv}{dt} = a &= -40 \\
v &= -40t + C \\
88 = v(0) &= 0 + C \\
C &= 88 \\
\frac{dr}{dt} = v &= 88 - 40t \\
r &= 88t - 20t^2 + K \\
0 = r(0) &= 0 + K \\
K &= 0 \\
r(t) &= 88t - 20t^2 \text{ or } 4t(22 - 5t)
\end{aligned}$$

When the car has stopped, its velocity is zero. Therefore $0 = v = 88 - 40t$, whence $t = \frac{88}{40} = \frac{22}{10} = 2.2$ s. At this time the car's position is $r(2.2) = 8.8(11) = 96.8$ ft.