

Spring 2006 Math 151

Exam 1A: Solutions

Mon, 20/Feb ©2006, Art Belmonte

1. (d) The angle between $\mathbf{v} = [-1, \sqrt{3}]$ and $\mathbf{w} = [1, \sqrt{3}]$ is

$$\begin{aligned}\theta &= \cos^{-1} \left(\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|} \right) \\ &= \cos^{-1} \left(\frac{-1 + 3}{\sqrt{1+3}\sqrt{1+3}} \right) \\ &= \cos^{-1} \frac{2}{4} = \cos^{-1} \frac{1}{2} = \frac{\pi}{3}.\end{aligned}$$

2. (b) With $g(x) = 2x^2 - 8x^{1/2} + 4x$, we have $g'(x) = 4x - 4x^{-1/2} + 4$, whence $g'(4) = 16 - 2 + 4 = 18$.

3. (d) The rational function

$$y = g(x) = \frac{x^2 - 1}{(x+1)(x+5)} = \frac{(x-1)(x+1)}{(x+1)(x+5)} = \frac{x-1}{x+5}$$

is defined for $x \neq -1, -5$.

- Since $\lim_{x \rightarrow \pm\infty} g(x) = \lim_{x \rightarrow \pm\infty} \frac{1 - \frac{1}{x}}{1 + \frac{5}{x}} = 1$, only $y = 1$ is a horizontal asymptote.

- Since

$$\lim_{x \rightarrow -5^-} g(x) = \lim_{x \rightarrow -5^-} \frac{x-1}{x+5} = \frac{-6}{0^-} = +\infty,$$

we see that $x = -5$ is a vertical asymptote.

- Since

$$\lim_{x \rightarrow -1} g(x) = \lim_{x \rightarrow -1} \frac{x-1}{x+5} = \frac{-2}{4} = -\frac{1}{2} \neq \pm\infty,$$

we see that $x = -1$ is *not* a vertical asymptote.

4. (e) The piecewise polynomial function

$$f(x) = \begin{cases} cx^2 - 1, & \text{if } x < -1 \\ 2c - x, & \text{if } x \geq -1 \end{cases}$$

is continuous for $x \neq -1$ since polynomials are continuous on \mathbb{R} . So the crux of the problem is to determine conditions on c such that $\lim_{x \rightarrow -1} f(x) = f(-1)$. Let's match the left and right limits of f at $x = -1$.

- $\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^-} (cx^2 - 1) = c - 1$

- $\lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^+} (2c - x) = 2c + 1$

- Set $c - 1 = 2c + 1$, whence $c = -2$. Thus

$$f(x) = \begin{cases} -2x^2 - 1, & \text{if } x < -1 \\ -4 - x, & \text{if } x \geq -1 \end{cases} \text{ and}$$

$$\lim_{x \rightarrow -1} f(x) = -3 = f(-1).$$

- Therefore, for $c = -2$, the piecewise function f is continuous at $x = -1$. Since f was already continuous for $x \neq -1$, we have that f is continuous on \mathbb{R} .

5. (b) Eliminate the parameter t in the parametric equations $x = 7 - 6t$ and $y = 8 + 2t$ to obtain the Cartesian form $y = mx + b$. This will allow us to read off the slope m of the line. Now $t = \frac{1}{6}(7 - x)$, whence

$$y = 8 + 2 \left(\frac{1}{6}(7 - x) \right) = -\frac{1}{3}x + \left(8 + \frac{7}{3} \right)$$

Hence the slope of the line is $m = -\frac{1}{3}$.

6. (b) With $f(x) = x^3 + 6x - 4$, we have $f(0) = -4$ and $f(1) = 3$. Since f , a polynomial, is continuous on \mathbb{R} , we conclude by the Intermediate Value Theorem (IVT) that f has a root in the interval $(0, 1)$; i.e., $f(c) = 0$ for some c such that $0 < c < 1$.
7. (a) The function $f(x) = \sqrt{x^2 - 6x + 5}$ is defined provided $x^2 - 6x + 5 = (x - 1)(x - 5) \geq 0$. This occurs for $x \leq 1$ or $x \geq 5$. In other words, the domain of f is $(-\infty, 1] \cup [5, \infty)$.

8. (a) We have

$$\lim_{x \rightarrow 5} \frac{x - 5}{x^2 - 8x + 15} = \lim_{x \rightarrow 5} \frac{(x - 5)}{(x - 5)(x - 3)} = \lim_{x \rightarrow 5} \frac{1}{x - 3} = \frac{1}{2}.$$

9. (d) Velocity is the derivative of position $s = \frac{t}{t^2 + 1}$. When $t = \sqrt{2}$ seconds, we have

$$\frac{ds}{dt} = \frac{(t^2 + 1)(1) - (t)(2t)}{(t^2 + 1)^2} = \frac{3 - 4}{3^2} = -\frac{1}{9} \text{ ft/s.}$$

10. (e) Use limit laws and the given information.

$$\lim_{x \rightarrow a} (f(x) + (g(x))^2) = 12$$

$$\lim_{x \rightarrow a} f(x) + \left(\lim_{x \rightarrow a} g(x) \right)^2 = 12$$

$$\lim_{x \rightarrow a} f(x) = 12 - (4)^2 = -4$$

11. (d) We have

$$\lim_{x \rightarrow -4^-} \frac{|x + 4|}{x^2 - 16} = \lim_{x \rightarrow -4^-} \frac{-(x + 4)}{(x + 4)(x - 4)} = \lim_{x \rightarrow -4^-} \frac{-1}{x - 4} = \frac{1}{8}.$$

12. (c) Use derivative rules, then substitute numerical data.

$$h'(x) = (xf(x) - g(x))'$$

$$h'(x) = (1)f(x) + xf'(x) - g'(x)$$

$$h'(2) = f(2) + 2f'(2) - g'(2)$$

$$h'(2) = 2 + 2(-1) - 4 = -4$$

13. (c) The instantaneous velocity is approximately equal to the average velocity nearby. Accordingly, at $t = 2$ sec we have

$$\frac{dh}{dt} \approx \frac{\Delta h}{\Delta t} = \frac{53 - 50}{2.1 - 2.0} = \frac{3}{0.1} = 30 \text{ m/s.}$$

14. The definition of the derivative involves a limit of a difference quotient. Take it step-by-step.

$$\begin{aligned} f'(2) &= \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{\frac{1}{x^2} - \frac{1}{4}}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{\left(\frac{4 - x^2}{4x^2}\right)}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{-(x - 2)(x + 2)}{4x^2(x - 2)} \\ &= \lim_{x \rightarrow 2} \frac{-(x + 2)}{4x^2} = \frac{-4}{16} = -\frac{1}{4} \end{aligned}$$

15. Given $\mathbf{r}(t) = [t^3 - t + 3, 8t^2]$ and $\mathbf{r}'(t) = [3t^2 - 1, 16t]$, here is a parametric representation of the tangent line to \mathbf{r} at $t = 2$.

$$\begin{aligned} \mathbf{L}(u) &= \mathbf{r}(2) + u\mathbf{r}'(2) \\ &= [9, 32] + u[11, 32] \\ &= [11u + 9, 32u + 32] \\ &= [x(u), y(u)]. \end{aligned}$$

(We used u as a parameter for the line so as not to confuse it with t , since t has the specific value 2. However it's fine if you used the conventional parameter t .)

16. Differentiate $y = \frac{3x - 2}{x^2 + 4}$ using the Quotient Rule to obtain $y' = \frac{(x^2 + 4)(3) - (3x - 2)(2x)}{(x^2 + 4)^2}$. The slope of the tangent line is $m = y'(1) = \frac{(5)(3) - (1)(2)}{25} = \frac{13}{25}$. A point on the tangent line is $(1, y(1)) = \left(1, \frac{1}{5}\right)$. The point-slope formula yields $y - \frac{1}{5} = \frac{13}{25}(x - 1)$ or $y = \frac{13}{25}x - \frac{8}{25}$.

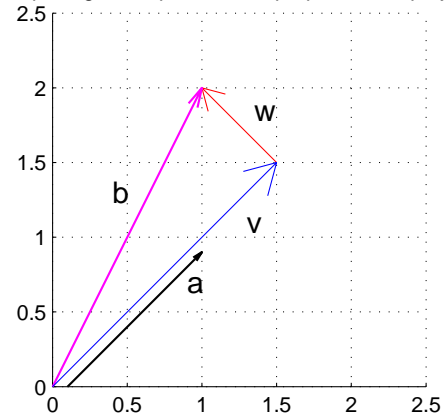
17. (a) The vector projection of $\mathbf{b} = [1, 2]$ onto $\mathbf{a} = [1, 1]$ is

$$\begin{aligned} \mathbf{v} = \text{proj}_{\mathbf{a}}\mathbf{b} &= \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|}\right) \frac{\mathbf{a}}{\|\mathbf{a}\|} \\ &= \left(\frac{1 + 2}{\sqrt{2}}\right) \frac{[1, 1]}{\sqrt{2}} = \left[\frac{3}{2}, \frac{3}{2}\right]. \end{aligned}$$

- (b) Let $\mathbf{w} = \text{orth}_{\mathbf{a}}\mathbf{b}$ be the perpendicular projection of the vector \mathbf{b} onto \mathbf{a} . Then $\mathbf{v} + \mathbf{w} = \text{proj}_{\mathbf{a}}\mathbf{b} + \text{orth}_{\mathbf{a}}\mathbf{b} = \mathbf{b}$. Accordingly, the distance we seek is $\|\mathbf{w}\|$. (See diagram at top of right column.)

$$\begin{aligned} \|\mathbf{w}\| &= \|\mathbf{b} - \mathbf{v}\| \\ &= \left\| [1, 2] - \left[\frac{3}{2}, \frac{3}{2}\right] \right\| \\ &= \left\| \left[-\frac{1}{2}, \frac{1}{2}\right] \right\| \\ &= \sqrt{\frac{1}{4} + \frac{1}{4}} = \frac{\sqrt{2}}{2} \end{aligned}$$

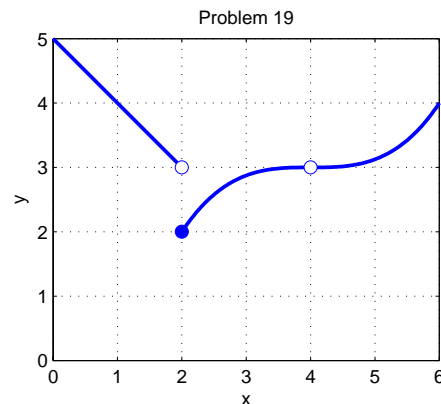
Decomposing \mathbf{b} into parallel and perpendicular projections



18. Multiply numerator and denominator by the conjugate of the numerator.

$$\begin{aligned} &\lim_{h \rightarrow 0} \frac{(\sqrt{16 + 2h} - 4)}{h} \cdot \frac{(\sqrt{16 + 2h} + 4)}{(\sqrt{16 + 2h} + 4)} \\ &= \lim_{h \rightarrow 0} \frac{(16 + 2h) - 16}{h(\sqrt{16 + 2h} + 4)} \\ &= \lim_{h \rightarrow 0} \frac{2h}{h(\sqrt{16 + 2h} + 4)} \\ &= \lim_{h \rightarrow 0} \frac{2}{\sqrt{16 + 2h} + 4} = \frac{2}{4 + 4} = \frac{1}{4} \end{aligned}$$

19. Here is a graph of the function $f(x)$.



- (a) Since $\lim_{x \rightarrow 4^-} f(x) = 3$ and $\lim_{x \rightarrow 4^+} f(x) = 3$, we deduce that $\lim_{x \rightarrow 4} f(x) = 3$.
- (b) Clearly f is not continuous at $x = 4$ since f is not defined at $x = 4$. (Note the hole in the graph of f thereat. The graph is broken; i.e., discontinuous.)
- (c) If f were differentiable at $x = 2$, then it would be continuous at $x = 2$ by a theorem in Section 3.1. But clearly f is not continuous at $x = 2$, since the graph is broken (discontinuous) there. [More precisely, $\lim_{x \rightarrow 2} f(x) \neq f(2)$ since the limit in question does not exist. See part (d).] Accordingly f is not differentiable at $x = 2$.
- (d) We have $\lim_{x \rightarrow 2^+} f(x) = 2$, whereas $\lim_{x \rightarrow 2^-} f(x) = 3$. (Thus $\lim_{x \rightarrow 2} f(x)$ does not exist since the left and right limits are not equal.)