Spring 2006 Math 151
Exam 1B: Solutions
Mon, 20/Feb

1. (d) Eliminate the parameter t in the parametric equations $x = 7 - 6t$ and $y = 8 + 2t$ to obtain the Cartesian form $y = mx + b$. This will allow us to read off the slope m of the line. Now $t = \frac{1}{6}(7 - x)$, whence
$$y = 8 + 2\left(\tfrac{1}{6}(7 - x)\right) = -\tfrac{1}{3}x + \left(8 + \tfrac{7}{3}\right)$$
Hence the slope of the line is $m = -\frac{1}{3}$.

2. (c) With $g(x) = 2x^2 - 8x^{1/2} + 4x$, we have $g'(x) = 4x - 4x^{-1/2} + 4$, whence $g'(4) = 16 - 2 + 4 = 18$.

3. (b) The rational function
$$y = g(x) = \frac{x^2 - 1}{(x+1)(x+5)} = \frac{(x-1)(x+1)}{(x+1)(x+5)} = \frac{x-1}{x+5}$$
is defined for $x \neq -1, -5$.
   - Since $\lim\limits_{x\to\pm\infty} g(x) = \lim\limits_{x\to\pm\infty} \dfrac{1 - \frac{1}{x}}{1 + \frac{5}{x}} = 1$, only $y = 1$ is a horizontal asymptote.
   - Since
   $$\lim_{x\to -5^-} g(x) = \lim_{x\to -5^-} \frac{x-1}{x+5} = \frac{-6}{0^-} = +\infty,$$
   we see that $x = -5$ is a vertical asymptote.
   - Since
   $$\lim_{x\to -1} g(x) = \lim_{x\to -1} \frac{x-1}{x+5} = \frac{-2}{4} = -\frac{1}{2} \neq \pm\infty,$$
   we see that $x = -1$ is *not* a vertical asymptote.

4. (a) Use derivative rules, then substitute numerical data.
$$\begin{aligned} h'(x) &= (xf(x) - g(x))' \\ h'(x) &= (1)\, f(x) + xf'(x) - g'(x) \\ h'(2) &= f(2) + 2f'(2) - g'(2) \\ h'(2) &= 2 + 2(-1) - 4 = -4 \end{aligned}$$

5. (d) Velocity is the derivative of position $s = \dfrac{t}{t^2 + 1}$. When $t = \sqrt{2}$ seconds, we have
$$\frac{ds}{dt} = \frac{\left(t^2+1\right)(1) - (t)(2t)}{\left(t^2+1\right)^2} = \frac{3-4}{3^2} = -\frac{1}{9} \text{ ft/s.}$$

6. (a) Use limit laws and the given information.
$$\begin{aligned} \lim_{x\to a}\left(f(x) + (g(x))^2\right) &= 12 \\ \lim_{x\to a} f(x) + \left(\lim_{x\to a} g(x)\right)^2 &= 12 \\ \lim_{x\to a} f(x) &= 12 - (4)^2 = -4 \end{aligned}$$

7. (b) We have
$$\lim_{x\to -4^-} \frac{|x+4|}{x^2 - 16} = \lim_{x\to -4^-} \frac{-(x+4)}{(x+4)(x-4)} = \lim_{x\to -4^-} \frac{-1}{x-4} = \frac{1}{8}.$$

8. (b) The function $f(x) = \sqrt{x^2 - 6x + 5}$ is defined provided $x^2 - 6x + 5 = (x-1)(x-5) \geq 0$. This occurs for $x \leq 1$ or $x \geq 5$. In other words, the domain of f is $(-\infty, 1] \cup [5, \infty)$.

9. (b) The piecewise polynomial function
$$f(x) = \begin{cases} cx^2 - 1, & \text{if } x < -1 \\ 2c - x, & \text{if } x \geq -1 \end{cases}$$
is continuous for $x \neq -1$ since polynomials are continuous on $\mathbb{R}$. So the crux of the problem is to determine conditions on c such that $\lim\limits_{x\to -1} f(x) = f(-1)$. Let's match the left and right limits of f at $x = -1$.
   - $\lim\limits_{x\to -1^-} f(x) = \lim\limits_{x\to -1^-} \left(cx^2 - 1\right) = c - 1$
   - $\lim\limits_{x\to -1^+} f(x) = \lim\limits_{x\to -1^+} (2c - x) = 2c + 1$
   - Set $c - 1 = 2c + 1$, whence $c = -2$. Thus $f(x) = \begin{cases} -2x^2 - 1, & \text{if } x < -1 \\ -4 - x, & \text{if } x \geq -1 \end{cases}$ and $\lim\limits_{x\to -1} f(x) = -3 = f(-1)$.
   - Therefore, for $c = -2$, the piecewise function f is continuous at $x = -1$. Since f was already continuous for $x \neq -1$, we have that f is continuous on $\mathbb{R}$.

10. (c) With $f(x) = x^3 + 6x - 4$, we have $f(0) = -4$ and $f(1) = 3$. Since f, a polynomial, is continuous on $\mathbb{R}$, we conclude by the Intermediate Value Theorem (IVT) that f has a root in the interval $(0, 1)$; i.e., $f(c) = 0$ for some c such that $0 < c < 1$.

11. (c) The angle between $\mathbf{v} = \left[-1, \sqrt{3}\right]$ and $\mathbf{w} = \left[1, \sqrt{3}\right]$ is
$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\,\|\mathbf{w}\|}\right) \\ &= \cos^{-1}\left(\frac{-1+3}{\sqrt{1+3}\sqrt{1+3}}\right) \\ &= \cos^{-1}\tfrac{2}{4} = \cos^{-1}\tfrac{1}{2} = \tfrac{\pi}{3}. \end{aligned}$$

12. (c) We have
$$\lim_{x\to 5} \frac{x-5}{x^2 - 8x + 15} = \lim_{x\to 5} \frac{(x-5)}{(x-5)(x-3)} = \lim_{x\to 5} \frac{1}{x-3} = \frac{1}{2}.$$

13. (b) The instantaneous velocity is approximately equal to the average velocity nearby. Accordingly, at $t = 2$ sec we have
$$\frac{dh}{dt} \approx \frac{\Delta h}{\Delta t} = \frac{53 - 50}{2.1 - 2.0} = \frac{3}{0.1} = 30 \text{ m/s.}$$

[The workout problems are on the next page.]

14. The definition of the derivative involves a limit of a difference quotient. Take it step-by-step.

$$\begin{aligned} f'(2) &= \lim_{x\to 2}\frac{f(x)-f(2)}{x-2}\\ &= \lim_{x\to 2}\frac{\dfrac{1}{x^2}-\dfrac{1}{4}}{x-2}\\ &= \lim_{x\to 2}\frac{\left(\dfrac{4-x^2}{4x^2}\right)}{x-2}\\ &= \lim_{x\to 2}\frac{-(x-2)(x+2)}{4x^2(x-2)}\\ &= \lim_{x\to 2}\frac{-(x+2)}{4x^2} = \frac{-4}{16} = -\frac{1}{4} \end{aligned}$$

15. Given $\mathbf{r}(t) = \left[t^3 - t + 3, 8t^2\right]$ and $\mathbf{r}'(t) = \left[3t^2 - 1, 16t\right]$, here is a parametric representation of the tangent line to $\mathbf{r}$ at $t = 2$.

$$\begin{aligned} \mathbf{L}(u) &= \mathbf{r}(2) + u\mathbf{r}'(2)\\ &= [9, 32] + u[11, 32]\\ &= [11u + 9, 32u + 32]\\ &= [x(u), y(u)]. \end{aligned}$$

(We used u as a parameter for the line so as not to confuse it with t, since t has the specific value 2. However it's fine if you used the conventional parameter t.)

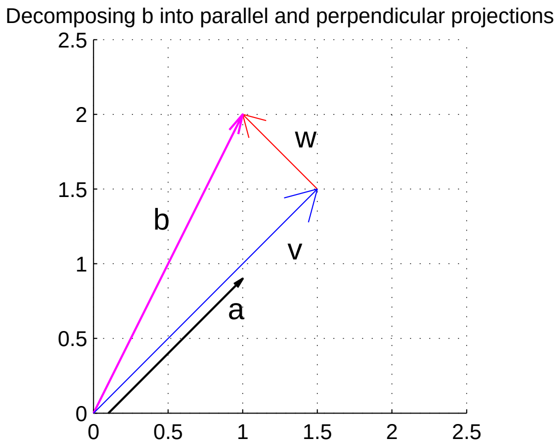
16. Differentiate $y = \dfrac{3x-2}{x^2+4}$ using the Quotient Rule to obtain $y' = \dfrac{(x^2+4)(3) - (3x-2)(2x)}{(x^2+4)^2}$. The slope of the tangent line is $m = y'(1) = \dfrac{(5)(3)-(1)(2)}{25} = \dfrac{13}{25}$. A point on the tangent line is $(1, y(1)) = \left(1, \frac{1}{5}\right)$. The point-slope formula yields $y - \frac{1}{5} = \frac{13}{25}(x-1)$ or $y = \frac{13}{25}x - \frac{8}{25}$.

17. (a) The vector projection of $\mathbf{b} = [1, 2]$ onto $\mathbf{a} = [1, 1]$ is

$$\begin{aligned} \mathbf{v} = \operatorname{proj}_{\mathbf{a}}\mathbf{b} &= \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}\right)\frac{\mathbf{a}}{\|\mathbf{a}\|}\\ &= \left(\frac{1+2}{\sqrt{2}}\right)\frac{[1,1]}{\sqrt{2}} = \left[\frac{3}{2}, \frac{3}{2}\right]. \end{aligned}$$

(b) Let $\mathbf{w} = \operatorname{orth}_{\mathbf{a}}\mathbf{b}$ be the perpendicular projection of the vector $\mathbf{b}$ onto $\mathbf{a}$. Then $\mathbf{v} + \mathbf{w} = \operatorname{proj}_{\mathbf{a}}\mathbf{b} + \operatorname{orth}_{\mathbf{a}}\mathbf{b} = \mathbf{b}$. Accordingly, the distance we seek is $\|\mathbf{w}\|$. (See diagram at top of right column.)

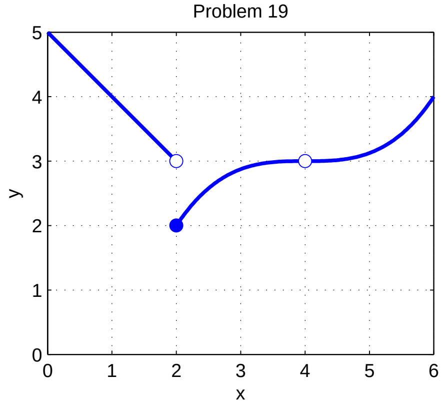
$$\begin{aligned} \|\mathbf{w}\| &= \|\mathbf{b} - \mathbf{v}\|\\ &= \left\|[1,2] - \left[\tfrac{3}{2}, \tfrac{3}{2}\right]\right\|\\ &= \left\|\left[-\tfrac{1}{2}, \tfrac{1}{2}\right]\right\|\\ &= \sqrt{\tfrac{1}{4} + \tfrac{1}{4}} = \frac{\sqrt{2}}{2} \end{aligned}$$



18. Multiply numerator and denominator by the conjugate of the numerator.

$$\begin{aligned} &\lim_{h\to 0}\frac{\left(\sqrt{16+2h}-4\right)}{h}\cdot\frac{\left(\sqrt{16+2h}+4\right)}{\left(\sqrt{16+2h}+4\right)}\\ =\ &\lim_{h\to 0}\frac{(16+2h)-16}{h\left(\sqrt{16+2h}+4\right)}\\ =\ &\lim_{h\to 0}\frac{2h}{h\left(\sqrt{16+2h}+4\right)}\\ =\ &\lim_{h\to 0}\frac{2}{\sqrt{16+2h}+4} = \frac{2}{4+4} = \frac{1}{4} \end{aligned}$$

19. Here is a graph of the function $f(x)$.



(a) Since $\lim_{x\to 4^-} f(x) = 3$ and $\lim_{x\to 4^+} f(x) = 3$, we deduce that $\lim_{x\to 4} f(x) = 3$.

(b) Clearly f is not continuous at $x = 4$ since f is not *defined* at $x = 4$. (Note the hole in the graph of f thereat. The graph is broken; i.e., discontinuous.)

(c) If f *were* differentiable at $x = 2$, then it would be continuous at $x = 2$ by a theorem in Section 3.1. But clearly f is not continuous at $x = 2$, since the graph is broken (discontinuous) there. [More precisely, $\lim_{x\to 2} f(x) \neq f(2)$ since the limit in question does not exist. See part (d).] Accordingly f is not differentiable at $x = 2$.

(d) We have $\lim_{x\to 2^+} f(x) = 2$, whereas $\lim_{x\to 2^-} f(x) = 3$. (Thus $\lim_{x\to 2} f(x)$ does not exist since the left and right limits are not equal.)