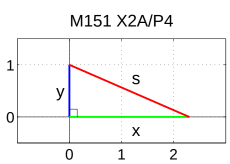
Fall 2006 Math 151
Exam 2A: Solutions
Mon, 30/Oct

1. (a) We have
$$\lim_{x\to 0}\frac{3x\cos x}{\sin 2x}=\lim_{x\to 0}\frac{\frac{3x}{2x}\cos x}{\frac{\sin 2x}{2x}}=\lim_{x\to 0}\frac{\frac{3}{2}\cos x}{\frac{\sin 2x}{2x}}=\frac{\frac{3}{2}(1)}{1}=\frac{3}{2}.$$

2. (b) The sine and cosine functions are continuous on $\mathbb{R}$. Moreover, compositions of continuous functions are continuous. Therefore,
$$\begin{aligned}\lim_{x\to 0}\cos\left(\sin(\sin x)\right) &= \cos\left(\sin(\sin 0)\right)\\ &= \cos\left(\sin 0\right)\\ &= \cos 0 = 1.\end{aligned}$$

3. (a) The slope of a horizontal tangent line is zero. Recall that $f(x)=2x-\sin 2x$. Thus
$$\begin{aligned}f'(x)=2-2\cos 2x &= 0\\ \cos 2x &= 1\\ 2x &= 2n\pi\\ x &= n\pi,\end{aligned}$$
where n is any integer.

4. (d) Use related rates. By the Pythagorean Theorem, we have $s^2=x^2+y^2=x^2+1$, whence
$$\begin{aligned}2s\frac{ds}{dt} &= 2x\frac{dx}{dt}\\ \frac{ds}{dt} &= \frac{x\frac{dx}{dt}}{s}=\frac{2x}{\sqrt{x^2+1}}.\end{aligned}$$



5. (d) Use properties of logarithms.
$$\begin{aligned}\log_3(3x+2) &= 2\\ 3x+2 &= 3^2=9\\ 3x &= 7\\ x &= 7/3\end{aligned}$$
Checking this answer in the original equation, we verify that $\log_3 9=2$.

6. (d) We have
$$\begin{aligned}h'(x) &= f'(g(x))\,g'(x)\\ h'(1) &= f'(g(1))\,g'(1)\\ h'(1) &= f'(3)\,g'(1)\\ h'(1) &= (3)(2)=6.\end{aligned}$$

7. (c) The range of f^{-1}, the inverse of $f(x)=\sqrt{1-x}$, is the *domain* of f. We require $1-x\geq 0$ or $1\geq x$; i.e., $(-\infty,1]$.

8. (c) If $f(x)=x\cos^2\left(x^3\right)$, the product and chain rules give $f'(x)=(1)\cos^2\left(x^3\right)+x\left(2\cos\left(x^3\right)\left(-\sin\left(x^3\right)\cdot 3x^2\right)\right)$; that is, $f'(x)=\cos^2\left(x^3\right)-6x^3\cos\left(x^3\right)\sin\left(x^3\right)$.

9. (a) We have $\mathbf{v}(t)=\mathbf{r}'(t)=\mathbf{i}+\left(\frac{\pi}{8}\cos\left(\frac{(t+1)\pi}{8}\right)\right)\mathbf{j}$, whence $\mathbf{v}(3)=\mathbf{i}$.

10. (a) Let $p(x)=ax^2+bx+c$. Then $p'(x)=2ax+b$ and $p''(x)=2a$. Now substitute the numerical data.
$$\begin{aligned}p(1)=a+b+c &= 1\\ p'(0)=b &= 2\\ p''(1)=2a &= 2\end{aligned}$$
This gives $a=1$, $b=2$, and $c=1-a-b=-2$. Hence $p(x)=x^2+2x-2$ and thus $p(-1)=1-2-2=-3$.

11. (e) Setting $\mathbf{r}(t)=\left[t^3-t,t^2-1\right]=[0,0]$, we deduce that $t=\pm 1$. Now $\mathbf{v}(t)=\mathbf{r}'(t)=\left[3t^2-1,2t\right]$. The relevant tangent vectors are $\mathbf{v}(1)=[2,2]$ and $\mathbf{v}(-1)=[2,-2]$. Observe that $\mathbf{v}_1\cdot\mathbf{v}_2=0$. This implies that the tangent vectors (and hence the tangent lines) are perpendicular. Hence the angle between them is 90° (graph on reverse).

12. (c) With $f(x)=x^{1/2}$, compute $f'(x)=\frac{1}{2}x^{-1/2}$. Then
$$\begin{aligned}L(x) &= f(100)+f'(100)(x-100)\\ L(90) &= 10+\tfrac{1}{20}(-10)=9.5.\end{aligned}$$

13. (b) The quadratic approximation of a quadratic function is said function! Now $p(x)=(x-1)^2$. So $q(1)=p(1)=0$. [Nonbelievers: grind out the details if you must.]

14. (e) Since $f(1)=2$ and $g=f^{-1}$, we have $g(2)=1$. Thus $g'(2)=\dfrac{1}{f'(g(2))}=\dfrac{1}{f'(1)}=\dfrac{1}{3}$.

15. Recall that $\mathbf{r}(t)=[x(t),y(t)]=\left[te^{-t},(2t+1)^{1/3}\right]$.

    (a) So $\mathbf{r}'(t)=\left[\frac{dx}{dt},\frac{dy}{dt}\right]=\left[1e^{-t}+t\left(-e^{-t}\right),\frac{1}{3}(2t+1)^{-2/3}\cdot 2\right]$.

    (b) Hence $m=\dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}=\dfrac{\frac{2}{3}(2t+1)^{-2/3}}{(1-t)e^{-t}}$.

    (c) At $(x,y)=(0,1)$, we have $t=0$ and $m=\frac{2}{3}$. The point-slope formula gives a Cartesian equation of the tangent line.
$$\begin{aligned}y-1 &= \tfrac{2}{3}(x-0)\\ y &= \tfrac{2}{3}x+1\end{aligned}$$
    Alternatively, a direction vector for the line is $\mathbf{r}'(0)=\left[1,\frac{2}{3}\right]$. A parametric representation of the tangent line is thus $\mathbf{L}(t)=[0,1]+t\left[1,\frac{2}{3}\right]$ or $\mathbf{L}(t)=\left[t,\frac{2}{3}t+1\right]$. (See graph on reverse.)

16. Implicitly differentiate $x^2 - xy + y^3 = 25$ with respect to x, then immediately substitute $(x, y) = (1, 3)$. This let's you do arithmetic instead of algebra (although the latter is fine too). Below, y'_* represents the value of dy/dx at $(1, 3)$.

$$\begin{aligned} 2x - \left(1y + xy'\right) + 3y^2 y' &= 0 \\ 2 - \left(3 + y'_*\right) + 27y'_* &= 0 \\ 26y'_* &= 1 \\ y'_* &= 1/26 \end{aligned}$$

Now use the point-slope formula.

$$\begin{aligned} y - 3 &= \tfrac{1}{26}(x - 1) \\ y &= \tfrac{1}{26}x + \tfrac{77}{26} \end{aligned}$$

See graph at lower right.

17. Recall the position function $\mathbf{r}(t) = (2\cos 2t)\,\mathbf{i} + (2\sin 2t)\,\mathbf{j}$.

(a) The velocity is

$$\mathbf{v}(t) = \mathbf{r}'(t) = (-4\sin 2t)\,\mathbf{i} + (4\cos 2t)\,\mathbf{j},$$

whence $\mathbf{v}(\pi/12) = -2\mathbf{i} + 2\sqrt{3}\,\mathbf{j}$.

(b) The speed at time t is

$$\|\mathbf{v}(t)\| = \sqrt{16\sin^2 2t + 16\cos^2 2t} = \sqrt{16} = 4.$$

(c) The acceleration is

$$\mathbf{a}(t) = \mathbf{v}'(t) = (-8\cos 2t)\,\mathbf{i} + (-8\sin 2t)\,\mathbf{j}.$$

The dot product of $\mathbf{v}(t)$ and $\mathbf{a}(t)$ is

$$32\sin 2t\cos 2t - 32\sin 2t\cos 2t = 0.$$

Hence $\mathbf{v}(t)$ and $\mathbf{a}(t)$ are orthogonal (perpendicular).

(d) Now $\mathbf{r}(t) = [x(t), y(t)] = [2\cos 2t, 2\sin 2t]$. Hence

$$x^2 + y^2 = 4\cos^2 2t + 4\sin^2 2t = 4 = 2^2.$$

As t increases, the circle $x^2 + y^2 = 2^2$ is traversed over and over again in a counterclockwise direction. (See graph at bottom right.)

18. With $f(x) = \dfrac{x}{(7-3x)^{1/2}}$, the derivative of f is

$$f'(x) = \frac{(7-3x)^{1/2}(1) - x\left(\frac{1}{2}(7-3x)^{-1/2}(-3)\right)}{7-3x}.$$

(a) Therefore,

$$f'(2) = \frac{(1)(1) - 2\left(\frac{1}{2}(1)(-3)\right)}{1} = \frac{1+3}{1} = 4.$$

(b) Note that $f(2) = 2$. Thus $f'\big(f(2)\big) = f'(2) = 4$.

(c) Recall that $h(x) = f(f(x))$. The Chain Rule gives

$$\begin{aligned} h'(x) &= f'(f(x))\,f'(x) \\ h'(2) &= f'(f(2))\,f'(2) = 4^2 = 16. \end{aligned}$$

(d) Now $h(2) = f(f(2)) = f(2) = 2$. Thus

$$L(x) = h(2) + h'(2)(x-2) = 2 + 16(x-2) = 16x - 30.$$

19. Let A, B, and H represent the area, base, and height of the triangle, respectively. Recall that $A = \frac{1}{2}BH$. At the instant of time t_* in question, we have $A = 100\text{ cm}^2$, $H = 10$ cm, and thus $B = 2A/H = 200/10 = 20$ cm. Use related rates. Differentiate with respect to t, then substitute the numerical data given in the problem.

$$\begin{aligned} \frac{dA}{dt} &= \frac{1}{2}\left(\frac{dB}{dt}H + B\frac{dH}{dt}\right) \\ -4 &= \frac{1}{2}\left(10B'(t_*) + 20(2)\right) \\ -4 &= 5B'(t_*) + 20 \\ -24 &= 5B'(t_*) \\ B'(t_*) &= -24/5 = -4.8\text{ cm/min} \end{aligned}$$

Illustrative graphs