Fall 2006 Math 151
Exam 2B: Solutions
Mon, 30/Oct ©2006, Art Belmonte

1. (e) Use properties of logarithms.

$$\begin{aligned} \log_3(3x+2) &= 2 \\ 3x+2 &= 3^2 = 9 \\ 3x &= 7 \\ x &= 7/3 \end{aligned}$$

Checking this answer in the original equation, we verify that $\log_3 9 = 2$.

2. (c) We have

$$\lim_{x\to 0} \frac{3x\cos x}{\sin 2x} = \lim_{x\to 0} \frac{\frac{3x}{2x}\cos x}{\frac{\sin 2x}{2x}} = \lim_{x\to 0} \frac{\frac{3}{2}\cos x}{\frac{\sin 2x}{2x}} = \frac{\frac{3}{2}(1)}{1} = \frac{3}{2}.$$

3. (b) With $f(x) = x^{1/2}$, compute $f'(x) = \frac{1}{2}x^{-1/2}$. Then

$$\begin{aligned} L(x) &= f(100) + f'(100)(x-100) \\ L(90) &= 10 + \tfrac{1}{20}(-10) = 9.5. \end{aligned}$$

4. (d) The slope of a horizontal tangent line is zero. Recall that $f(x) = 2x - \sin 2x$. Thus

$$\begin{aligned} f'(x) = 2 - 2\cos 2x &= 0 \\ \cos 2x &= 1 \\ 2x &= 2n\pi \\ x &= n\pi, \end{aligned}$$

where n is any integer.

5. (a) Since $f(1) = 2$ and $g = f^{-1}$, we have $g(2) = 1$. Thus $g'(2) = \dfrac{1}{f'(g(2))} = \dfrac{1}{f'(1)} = \dfrac{1}{3}$.

6. (d) The sine and cosine functions are continuous on $\mathbb{R}$. Moreover, compositions of continuous functions are continuous. Therefore,

$$\begin{aligned} \lim_{x\to 0} \cos\left(\sin(\sin x)\right) &= \cos\left(\sin(\sin 0)\right) \\ &= \cos\left(\sin 0\right) \\ &= \cos 0 = 1. \end{aligned}$$

7. (a) Let $p(x) = ax^2 + bx + c$. Then $p'(x) = 2ax + b$ and $p''(x) = 2a$. Now substitute the numerical data.

$$\begin{aligned} p(1) = a + b + c &= 1 \\ p'(0) = b &= 2 \\ p''(1) = 2a &= 2 \end{aligned}$$

This gives $a = 1$, $b = 2$, and $c = 1 - a - b = -2$. Hence $p(x) = x^2 + 2x - 2$ and thus $p(-1) = 1 - 2 - 2 = -3$.

8. (c) Setting $\mathbf{r}(t) = \left[t^3 - t, t^2 - 1\right] = [0, 0]$, we deduce that $t = \pm 1$. Now $\mathbf{v}(t) = \mathbf{r}'(t) = \left[3t^2 - 1, 2t\right]$. The relevant tangent vectors are $\mathbf{v}(1) = [2, 2]$ and $\mathbf{v}(-1) = [2, -2]$. Observe that $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$. This implies that the tangent vectors (and hence the tangent lines) are perpendicular. Hence the angle between them is 90° (graph on reverse).

9. (e) The quadratic approximation of a quadratic function is said function! Now $p(x) = (x-1)^2$. So $q(1) = p(1) = 0$. [Nonbelievers: grind out the details if you must.]

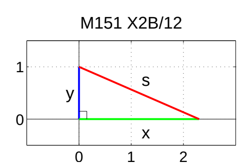
10. (b) We have

$$\begin{aligned} h'(x) &= f'(g(x))\,g'(x) \\ h'(1) &= f'(g(1))\,g'(1) \\ h'(1) &= f'(3)\,g'(1) \\ h'(1) &= (3)(2) = 6. \end{aligned}$$

11. (c) We have $\mathbf{v}(t) = \mathbf{r}'(t) = \mathbf{i} + \left(\frac{\pi}{8}\cos\left(\frac{(t+1)\pi}{8}\right)\right)\mathbf{j}$, whence $\mathbf{v}(3) = \mathbf{i}$.

12. (d) Use related rates. By the Pythagorean Theorem, we have $s^2 = x^2 + y^2 = x^2 + 1$, whence

$$\begin{aligned} 2s\frac{ds}{dt} &= 2x\frac{dx}{dt} \\ \frac{ds}{dt} &= \frac{x\frac{dx}{dt}}{s} = \frac{2x}{\sqrt{x^2+1}}. \end{aligned}$$



13. (a) The range of f^{-1}, the inverse of $f(x) = \sqrt{1-x}$, is the *domain* of f. We require $1 - x \geq 0$ or $1 \geq x$; i.e., $(-\infty, 1]$.

14. (e) If $f(x) = x\cos^2\left(x^3\right)$, the product and chain rules give $f'(x) = (1)\cos^2\left(x^3\right) + x\left(2\cos\left(x^3\right)\left(-\sin\left(x^3\right)\cdot 3x^2\right)\right)$; that is, $f'(x) = \cos^2\left(x^3\right) - 6x^3\cos\left(x^3\right)\sin\left(x^3\right)$.

15. Recall that $\mathbf{r}(t) = [x(t), y(t)] = \left[te^{-t}, (3t+1)^{1/5}\right]$.

 (a) So $\mathbf{r}'(t) = \left[\frac{dx}{dt}, \frac{dy}{dt}\right] = \left[1e^{-t} + t\left(-e^{-t}\right), \frac{1}{5}(3t+1)^{-4/5}\cdot 3\right]$.

 (b) Hence $m = \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{\frac{3}{5}(3t+1)^{-4/5}}{(1-t)e^{-t}}$.

 (c) At $(x, y) = (0, 1)$, we have $t = 0$ and $m = \frac{3}{5}$. The point-slope formula gives a Cartesian equation of the tangent line.

$$\begin{aligned} y - 1 &= \tfrac{3}{5}(x - 0) \\ y &= \tfrac{3}{5}x + 1 \end{aligned}$$

Alternatively, a direction vector for the line is $\mathbf{r}'(0) = \left[1, \frac{3}{5}\right]$. A parametric representation of the tangent line is thus $\mathbf{L}(t) = [0, 1] + t\left[1, \frac{3}{5}\right]$ or $\mathbf{L}(t) = \left[t, \frac{3}{5}t + 1\right]$. (See graph at bottom.)

16. Implicitly differentiate $x^2 - xy + y^3 = 25$ with respect to x, then immediately substitute $(x, y) = (2, 3)$. This let's you do arithmetic instead of algebra (although the latter is fine too). Below, y'_* represents the value of dy/dx at $(2, 3)$.

$$\begin{aligned} 2x - (1y + xy') + 3y^2y' &= 0 \\ 4 - (3 + 2y'_*) + 27y'_* &= 0 \\ 25y'_* &= -1 \\ y'_* &= -1/25 \end{aligned}$$

Now use the point-slope formula.

$$\begin{aligned} y - 3 &= -\tfrac{1}{25}(x - 2) \\ y &= -\tfrac{1}{25}x + \tfrac{77}{25} \end{aligned}$$

See graph at lower right.

17. Recall the position function $\mathbf{r}(t) = (2\cos 4t)\,\mathbf{i} + (2\sin 4t)\,\mathbf{j}$.

(a) The velocity is

$$\mathbf{v}(t) = \mathbf{r}'(t) = (-8\sin 4t)\,\mathbf{i} + (8\cos 4t)\,\mathbf{j},$$

whence $\mathbf{v}(\pi/12) = -4\sqrt{3}\mathbf{i} + 4\mathbf{j}$.

(b) The speed at time t is

$$\|\mathbf{v}(t)\| = \sqrt{64\sin^2 4t + 64\cos^2 4t} = \sqrt{64} = 8.$$

(c) The acceleration is

$$\mathbf{a}(t) = \mathbf{v}'(t) = (-32\cos 4t)\,\mathbf{i} + (-32\sin 4t)\,\mathbf{j}.$$

The dot product of $\mathbf{v}(t)$ and $\mathbf{a}(t)$ is

$$256\sin 4t\cos 4t - 256\sin 4t\cos 4t = 0.$$

Hence $\mathbf{v}(t)$ and $\mathbf{a}(t)$ are orthogonal (perpendicular).

(d) Now $\mathbf{r}(t) = [x(t), y(t)] = [2\cos 4t, 2\sin 4t]$. Hence

$$x^2 + y^2 = 4\cos^2 4t + 4\sin^2 4t = 4 = 2^2.$$

As t increases, the circle $x^2 + y^2 = 2^2$ is traversed over and over again in a counterclockwise direction. (See graph at bottom right.)

18. With $f(x) = \dfrac{x}{(7-3x)^{1/2}}$, the derivative of f is

$$f'(x) = \frac{(7-3x)^{1/2}(1) - x\left(\frac{1}{2}(7-3x)^{-1/2}(-3)\right)}{7-3x}.$$

(a) Therefore,

$$f'(0) = \frac{\left(\sqrt{7}\right)(1) - 0\left(\frac{1}{2}\left(\frac{1}{\sqrt{7}}\right)(-3)\right)}{7} = \frac{\sqrt{7}}{7}.$$

(b) Note that $f(0) = 0$. Thus $f'\left(f(0)\right) = f'(0) = \dfrac{\sqrt{7}}{7}$.

(c) Recall that $h(x) = f(f(x))$. The Chain Rule gives

$$\begin{aligned} h'(x) &= f'(f(x))\,f'(x) \\ h'(0) &= f'(f(0))\,f'(0) = \left(\frac{\sqrt{7}}{7}\right)^2 = \frac{1}{7}. \end{aligned}$$

(d) Now $h(0) = f(f(0)) = f(0) = 0$. Thus

$$L(x) = h(0) + h'(0)(x - 0) = 0 + \tfrac{1}{7}(x - 0) = \tfrac{1}{7}x.$$

19. Let A, B, and H represent the area, base, and height of the triangle, respectively. Recall that $A = \frac{1}{2}BH$. At the instant of time t_* in question, we have $A = 300\text{ cm}^2$, $B = 15$ cm, and thus $H = 2A/B = 600/15 = 40$ cm. Use related rates. Differentiate with respect to t, then substitute the numerical data given in the problem.

$$\begin{aligned} \frac{dA}{dt} &= \frac{1}{2}\left(\frac{dB}{dt}H + B\frac{dH}{dt}\right) \\ A'(t_*) &= \frac{1}{2}(20(40) + 15(-4)) \\ &= \frac{1}{2}(800 - 60) \\ &= \frac{1}{2}(740) \\ &= 370\text{ cm}^2/\text{min} \end{aligned}$$

Illustrative graphs