

Fall 2006 Math 151

Exam 3B: Solutions

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- (e) We have $\frac{d}{dx} (\tan^{-1}(\sin x)) = \frac{1}{1 + (\sin x)^2} \cdot \cos x$.
- (d) Let y be the temperature of the turkey at time t . Using the law of exponential growth/decay, we have that $dy/dt = -\frac{1}{2}y$ implies $y = y_0 e^{-t/2} = 57e^{-t/2}$. Thus $y(\ln 9) = 57e^{-\frac{1}{2} \ln 9} = 57e^{\ln \frac{1}{3}} = 57 \left(\frac{1}{3}\right) = 19^\circ\text{C}$.
- (c) Since $\frac{d}{dx} \left(\frac{1}{4}x^4 + 2x^3 + 9\right) = x^3 + 6x^2$, we conclude that $F(x) = \frac{1}{4}x^4 + 2x^3 + 9$ is an antiderivative of the function $f(x) = x^3 + 6x^2$.
- (a) This is an indeterminate power, 1^∞ .

$$y = (\cos x)^{1/x^2}$$

$$\ln y = \frac{\ln(\cos x)}{x^2}$$

$$\lim_{x \rightarrow 0} \ln y \stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0} \frac{\frac{-\sin x}{\cos x}}{2x} = \lim_{x \rightarrow 0} \frac{-\tan x}{2x}$$

$$\lim_{x \rightarrow 0} \ln y \stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0} \frac{-\sec^2 x}{2} = -\frac{1}{2}$$

Therefore, $\lim y = \lim e^{\ln y} = e^{-1/2}$.

- (e) The sum $\sum_{n=3}^{2006} \frac{1}{\sqrt{n}}$ fits the bill.
- (b) Let $\theta = \tan^{-1} 2$. Draw a right triangle. Via trigonometry, we see that $\cos \theta = 1/\sqrt{5}$. (See picture on reverse.)
- (c) Recall that $y = e^{-x^2/2}$. Find where $y'' < 0$.

$$y' = e^{-x^2/2} (-x) = -xe^{-x^2/2}$$

$$y'' = -\left(e^{-x^2/2} + xe^{-x^2/2} (-x)\right)$$

$$y'' = (x^2 - 1)e^{-x^2/2}$$

Since $e^{-x^2/2} > 0$ for $x \in \mathbb{R}$, we see that $y'' < 0$ where $x^2 - 1 < 0$; i.e., $x^2 < 1$ or $-1 < x < 1$.

- (b) We have

$$\lim_{x \rightarrow 2} \frac{x^3 - 12x + 16}{x^3 - 3x^2 + 4} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow 2} \frac{3x^2 - 12}{3x^2 - 6x}$$

$$\stackrel{\text{L'H}}{=} \lim_{x \rightarrow 2} \frac{6x}{6x - 6} = \frac{12}{6} = 2.$$

- (e) Using properties of summation, we have

$$\sum (5f + 2g) = 5 \sum f + 2 \sum g = 5(101) + 2(35) = 575.$$

- (b) Recall $f(x) = (x^2 - 6x + 10)^{-1}$. Thus

$$f'(x) = -\frac{2x - 6}{(x^2 - 6x + 10)^2} \text{ for all } x \in \mathbb{R} \text{ since}$$

$$x^2 - 6x + 10 = x^2 - 6x + 9 + 1 = (x - 3)^2 + 1 > 0.$$

Hence $f'(x) = 0$ implies $2x - 6 = 0$; i.e., $x = 3 \in [0, 4]$ is the lone critical number of f . Crank out function values at critical numbers and endpoints: $f(0) = 1/10$, $f(3) = 1$, $f(4) = 1/2$. We conclude that the absolute maximum value of f on $[0, 4]$ is $f(3) = 1$.

- (e) Use logarithmic differentiation.

$$y = (1+x)^x$$

$$\ln y = x \ln(1+x)$$

$$\frac{1}{y} \frac{dy}{dx} = (1) \ln(1+x) + x \left(\frac{1}{1+x}\right)$$

$$\frac{dy}{dx} = (1+x)^x \left(\ln(1+x) + \frac{x}{1+x}\right)$$

$$\frac{dy}{dx} = (1+x)^x \ln(1+x) + x(1+x)^{x-1}$$

- (b) The integral in question represents the area under f and above the x -axis. Just add areas of geometrical shapes.

$$(3 \times 5) + (3 \times 1) + \frac{1}{2}(3 \times 4) = 15 + 3 + 6 = 24$$

- (e) Curve 1 is f'' ; curve 2 is f ; curve 3 is f' .

- Change this indeterminate difference into an indeterminate quotient, then apply l'Hospital's Rule—twice.

$$\lim_{x \rightarrow 0} \left(\frac{1}{x} - \frac{1}{\ln(1+x)}\right) = \lim_{x \rightarrow 0} \frac{\ln(1+x) - x}{x \ln(1+x)}$$

$$\stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0} \frac{(1+x)^{-1} - 1}{(1) \ln(1+x) + \frac{x}{1+x}}$$

$$\stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0} \frac{-(1+x)^{-2}}{\frac{1}{1+x} + \frac{(1+x)(1) - x(1)}{(1+x)^2}}$$

$$= \frac{-1}{1+1} = -\frac{1}{2}$$

- (a) Now $y = 8 \ln(1+x) + \frac{1}{2} \ln x - \frac{1}{6} \ln(1+2x)$, whence

$$y' = \frac{8}{1+x} + \frac{1}{2x} - \frac{1}{6(1+2x)}.$$

- (b) With $f(x) = e^{3x} + (3x)^e + e^\pi + (3+e)^x$, we have $f'(x) = 3e^{3x} + e(3x)^{e-1}(3) + (3+e)^x \ln(3+e)$.

- (c) Since $g(x) = \ln|x| + \ln(\ln(x^3)) + \log_5(x^2 + 4)$,

$$\text{we have } g'(x) = \frac{1}{x} + \frac{1}{\ln(x^3)} \cdot \frac{3}{x} + \frac{2x}{(x^2 + 4) \ln 5}.$$

- The Mean Value Theorem states that if f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there exists a value $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$. [In other words, the slope of the tangent line to f at c equals the slope of the secant line through $(a, f(a))$ and $(b, f(b))$.]

17. • First determine the velocity function.

$$\begin{aligned} \mathbf{v}'(t) = \mathbf{a}(t) &= [1, t] \\ \mathbf{v}(t) &= \left[t, \frac{1}{2}t^2 \right] + \mathbf{C} \\ [1, -1] = \mathbf{v}(0) &= [0, 0] + \mathbf{C} \\ \mathbf{C} &= [1, -1] - [0, 0] = [1, -1] \\ \mathbf{v}(t) &= \left[t + 1, \frac{1}{2}t^2 - 1 \right] \end{aligned}$$

- Then determine the position function.

$$\begin{aligned} \mathbf{r}'(t) = \mathbf{v}(t) &= \left[t + 1, \frac{1}{2}t^2 - 1 \right] \\ \mathbf{r}(t) &= \left[\frac{1}{2}t^2 + t, \frac{1}{6}t^3 - t \right] + \mathbf{K} \\ [3, 1] = \mathbf{r}(1) &= \left[\frac{3}{2}, -\frac{5}{6} \right] + \mathbf{K} \\ \mathbf{K} &= [3, 1] - \left[\frac{3}{2}, -\frac{5}{6} \right] = \left[\frac{3}{2}, \frac{11}{6} \right] \\ \mathbf{r}(t) &= \left[\frac{1}{2}t^2 + t + \frac{3}{2}, \frac{1}{6}t^3 - t + \frac{11}{6} \right] \end{aligned}$$

18. Recall that $y = f(x) = \cos^{-1} x$.

- (a) The domain of f is $[-1, 1]$ or $-1 \leq x \leq 1$.
 (b) The range of f is $[0, \pi]$ or $0 \leq y \leq \pi$.
 (c) We have $\cos^{-1} \left(\cos \frac{3}{2}\pi \right) = \cos^{-1} (0) = \frac{\pi}{2}$.
 (d) Since $f'(x) = -1/\sqrt{1-x^2}$, we see that $f'(0) = -1$.
 (e) Via trig, $\sin \left(\cos^{-1} \frac{1}{4} \right) = \sqrt{15}/4$. (Picture at bottom.)
 (f) Finally, $\cos \left(\cos^{-1} \frac{1}{4} \right) = \frac{1}{4}$.

19. Recall that $f(x) = x^3 - 3x^2 - 45x - 22$.

- (a) The critical numbers of f occur where the derivative of f is zero or is undefined. Since $f'(x) = 3x^2 - 6x - 45 = 3(x^2 - 2x - 15) = 3(x+3)(x-5)$ is defined for all x , the critical numbers of f are $x = -3, 5$, where the derivative is zero.
 (b) • The point $(-3, f(-3))$ is a local maximum of f by the First Derivative Test since the sign of f' changes from $+$ to $-$ as x increases through -3 . (Alternatively, use the Second Derivative Test. Now $f''(x) = 6x - 6$, so $f''(-3) = -24 < 0$. This signifies a local maximum.)
 • The point $(5, f(5))$ is a local minimum of f by the First Derivative Test since the sign of f' changes from $-$ to $+$ as x increases through 5 . (Alternatively, use the Second Derivative Test. Now $f''(5) = 24 > 0$, which signifies a local minimum.)
 (c) An inflection point of f occurs where the second derivative changes sign. Now $f''(x) = 6x - 6 = 0$ implies $x = 1$. Note that f'' changes sign from $-$ to $+$ as x increases through 1 . Therefore, $(1, f(1))$ is an inflection point of f . [Graph of f is at right.→]

- (d) The Extreme Value Theorem guarantees that the continuous function f attains an absolute maximum on the closed interval $[-1, 0]$. From part (a), we see that there are no critical numbers of f in this interval. Accordingly, the absolute maximum must occur at an endpoint of the interval. Now $f(-1) = 19$, whereas $f(0) = -22$. Hence the absolute maximum of f on $[-1, 0]$ is 19 , which occurs at $x = -1$.

20. Place the house at the origin of the xy -plane. Let t be the time in hours after the couple has started walking. The husband's position is $[0, y] = [0, 4t - 10]$, whereas that of the wife is $[x, 0] = [3t, 0]$. By the Pythagorean Theorem, the distance between them is $z = \sqrt{x^2 + y^2} = (x^2 + y^2)^{1/2}$. [NOTE: This distance z is always positive since otherwise $x = y = 0$, whence $3t = 0$ and $4t - 10 = 0$, from which $0 = t = \frac{5}{2}$, a contradiction.]

- (a) To find the time they are closest together, solve $dz/dt = 0$, and verify that this gives the absolute minimum distance.

$$\begin{aligned} \frac{dz}{dt} &= 0 \\ \frac{1}{2} (x^2 + y^2)^{-1/2} \left(2x \frac{dx}{dt} + 2y \frac{dy}{dt} \right) &= 0 \\ \frac{x \frac{dx}{dt} + y \frac{dy}{dt}}{\sqrt{x^2 + y^2}} &= 0 \\ \left(x \frac{dx}{dt} + y \frac{dy}{dt} \right) / z &= 0 \\ ((3t)(3) + (4t - 10)(4)) / z &= 0 \\ dz/dt = (25t - 40) / z(t) &= 0 \\ t &= \frac{8}{5} \text{ hr} \end{aligned}$$

Note that $dz/dt < 0$ for $t < \frac{8}{5}$, whereas $dz/dt > 0$ for $t > \frac{8}{5}$. The First Derivative Test for Absolute Extrema allows us to conclude that $\left(\frac{8}{5}, z \left(\frac{8}{5} \right) \right)$ is the absolute minimum; i.e., the couple are closest together when $t = \frac{8}{5}$ hr or 96 minutes (1 hour, 36 minutes).

- (b) Recall that the distance between the couple at time t is $z = \sqrt{x^2 + y^2} = \sqrt{(3t)^2 + (4t - 10)^2} = \sqrt{25t^2 - 80t + 100}$.

At time $t = \frac{8}{5}$, we have

$$z = \sqrt{64 - 128 + 100} = \sqrt{36} = 6 \text{ miles.}$$

This is the closest distance between the couple.

