Fall 2007 Math 151
Exam 2A: Solutions
Wed, 31/Oct ©2007 Art Belmonte

1. (c) The derivative is $y' = \sec x \tan x + 2\sin x$, whence $y'\left(\frac{\pi}{3}\right) = 2\sqrt{3} + \sqrt{3} = 3\sqrt{3}$.

2. (e) Recall that $H(x) = g(f(x))$. Apply the Chain Rule.

$$\begin{aligned} H'(x) &= g'(f(x))\, f'(x) \\ H'(1) &= g'(f(1))\, f'(1) \\ H'(1) &= g'(3)\, f'(1) \\ H'(1) &= (9)(4) = 36 \end{aligned}$$

3. (c) Substitution gives $\lim_{\theta\to 0} \frac{\sin(\cos\theta)}{\sec\theta} = \frac{\sin(1)}{1} = \sin 1$.

4. (c) Now $f'(x) = \frac{1}{2}\left(1 + xe^{-2x}\right)^{-1/2}\left(e^{-2x} - 2xe^{-2x}\right)$ whence $f'(0) = \frac{1}{2}$.

5. (b) Recall $x = \cos t + \cos 2t$ and $y = \sin t + \sin 2t$. For $t = \frac{\pi}{2}$, we have $(x, y) = (-1, 1)$, a point on the tangent line. The slope of the tangent line for $t = \pi/2$ is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t + 2\cos 2t}{-\sin t - 2\sin 2t} = \frac{-2}{-1} = 2.$$

The point-slope formula then yields

$$\begin{aligned} y - 1 &= 2(x - (-1)) \\ y &= 2x + 3. \end{aligned}$$

6. (b) The area of a rectangle is $A = LW$, where L is its length and W its width. At the stated instant we have

$$\frac{dA}{dt} = \frac{dL}{dt}W + L\frac{dW}{dt} = (8)(10) + (20)(3) = 140\ \text{cm}^2/\text{s}.$$

7. (b) Now $y^2 = 1 + x^3$, whence $2y\frac{dy}{dt} = 3x^2\frac{dx}{dt}$ and thus $\frac{dx}{dt} = \frac{2y\frac{dy}{dt}}{3x^2} = \frac{2(3)(4)}{3(2)^2} = 2$ cm/s. (You may alternatively differentiate directly instead of implicitly.)

8. (d) Let $f(x) = x^{1/3}$. Then $f'(x) = \frac{1}{3}x^{-2/3}$. Thus $f(64) = 4$ and $f'(64) = \frac{1}{48}$. The linear approximation is $L(x) = f(64) + f'(64)(x - 64) = 4 + \frac{1}{48}(x - 64)$. Hence $\sqrt[3]{70} = f(70) \approx L(70) = 4\frac{1}{8}$.

9. (e) With $f(x) = \cos(x + \pi)$, we differentiate to obtain $f'(x) = -\sin(x + \pi)$ and $f''(x) = -\cos(x + \pi)$. Thus $f(0) = -1$, $f'(0) = 0$, and $f''(0) = 1$. The quadratic approximation is

$$\begin{aligned} Q(x) &= f(0) + f'(0)(x - 0) + \tfrac{1}{2}f''(0)(x - 0)^2 \\ Q(x) &= -1 + \tfrac{1}{2}x^2. \end{aligned}$$

10. (d) With $f(x) = e^x - 4x - \sin x$, let

$$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{e^x - 4x - \sin x}{e^x - 4 - \cos x}.$$

Then $x_2 = g(x_1) = g(0) = 0 - \frac{1 - 0 - 0}{1 - 4 - 1} = \frac{1}{4}$.

11. (c) Compute the first two derivatives of $p = ax^2 + bx + c$.

$$\begin{aligned} p'(x) &= 2ax + b \\ p''(x) &= 2a. \end{aligned}$$

The specified data give rise to three equations in the three unknowns a, b, and c.

$$\begin{aligned} 8 = p(1) &= a + b + c \\ 4 = p'(1) &= 2a + b \\ 6 = p''(1) &= 2a \end{aligned}$$

The third equation yields $a = 3$. Accordingly, we have $b = 4 - 2a = -2$ and $c = 8 - a - b = 8 - 3 + 2 = 7$. Thus $p(x) = 3x^2 - 2x + 7$. Hence $p\left(\frac{1}{2}\right) = 6\frac{3}{4}$.

12. (d) Recall $f(x) = 3 + x^2 + \tan(\pi x/2)$, $-1 < x < 1$, and $g(x) = f^{-1}(x)$. Since $f(0) = 3$, we have $g(3) = 0$. Now $f'(x) = 2x + \frac{\pi}{2}\sec^2(\pi x/2)$ and therefore $g'(3) = \frac{1}{f'(g(3))} = \frac{1}{f'(0)} = \frac{1}{\pi/2} = \frac{2}{\pi}$.

13. (a) As $x \to 3^+$, we have $\frac{x}{x-3} \to \frac{3}{0^+} = \infty$, whence $\left(\frac{1}{2}\right)^{\frac{x}{x-3}} \to 0$.

14. (d) Isolate x step-by-step, given that $y = \frac{1 - \sqrt{x}}{1 + \sqrt{x}}$.

$$\begin{aligned} y + y\sqrt{x} &= 1 - \sqrt{x} \\ (1 + y)\sqrt{x} &= 1 - y \\ \sqrt{x} &= \frac{1 - y}{1 + y} \\ x &= \left(\frac{1 - y}{1 + y}\right)^2 = \frac{(1 - y)^2}{(1 + y)^2} \end{aligned}$$

15. Get the slope of the tangent line via implicit differentiation.

$$\begin{aligned} y^4 - 4y^2 &= x^4 - 5x^2 \\ 4y^3\frac{dy}{dx} - 8y\frac{dy}{dx} &= 4x^3 - 10x \\ \frac{dy}{dx} &= \frac{2x^3 - 5x}{2y^3 - 4y} \\ \left.\frac{dy}{dx}\right|_{(\sqrt{5},2)} &= \frac{5\sqrt{5}}{8} \end{aligned}$$

[continued]

Now use the point-slope formula and finish it off.

$$\begin{aligned} y - 2 &= \frac{5\sqrt{5}}{8}\left(x - \sqrt{5}\right) \\ y &= \frac{5\sqrt{5}}{8}x + 2 - \frac{25}{8} \\ y &= \frac{5\sqrt{5}}{8}x - \frac{9}{8} \end{aligned}$$

16. Velocity is the derivative of position, acceleration is the derivative of velocity, and speed is the magnitude of velocity.

$$\begin{aligned} \mathbf{r}(t) &= [t\cos t, \quad t\sin t] \\ \mathbf{v} = \mathbf{r}'(t) &= [\cos t - t\sin t, \quad \sin t + t\cos t] \\ \mathbf{a} = \mathbf{v}'(t) &= [-\sin t - \sin t - t\cos t, \quad \cos t + \cos t - t\sin t] \end{aligned}$$

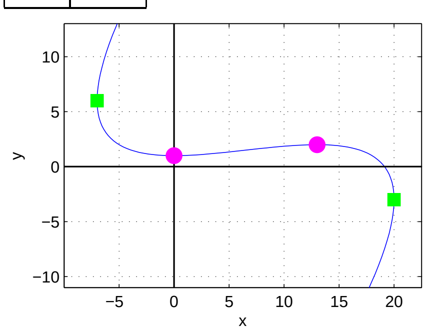
At $t = \pi$, we have

$$\begin{aligned} \text{position} &= \mathbf{r}(\pi) = [-\pi, 0] \\ \text{velocity} &= \mathbf{v}(\pi) = [-1, -\pi] \\ \text{acceleration} &= \mathbf{a}(\pi) = [\pi, -2] \\ \text{speed} &= \|\mathbf{v}(\pi)\| = \sqrt{1 + \pi^2} \end{aligned}$$

17. Recall that $x = 2t^3 + 3t^2 - 12t$ and $y = 2t^3 + 3t^2 + 1$.

- The tangent line to the curve will be *horizontal* where $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = 0$. This can occur if $\frac{dy}{dt} = 0$ & $\frac{dx}{dt} \neq 0$ simultaneously. Now $0 = \frac{dy}{dt} = 6t^2 + 6t = 6t\,(t+1)$ implies $t = -1, 0$. Observe that for these values of t we have $\frac{dx}{dt} = 6t^2 + 6t - 12 \neq 0$.
- The tangent line to the curve will be *vertical* where $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} =$ "$\pm\infty$", colloquially speaking. More precisely, this can occur where $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$ simultaneously. Now $0 = \frac{dx}{dt} = 6t^2 + 6t - 12$ implies $0 = 6\left(t^2 + t - 2\right) = 6\,(t-1)\,(t+2)$ and thus $t = -2, 1$, at which $\frac{dy}{dt} = 6t^2 + 6t \neq 0$.
- Here is the requested table and an illustrative plot. Recall that H = horizontal and V = vertical.

t	*type*
-2	V
-1	H
0	H
1	V



18. Now $C = 2\pi r$ implies $dC = 2\pi\, dr$. It was stated that $C = 20$ and $dC = 1$, with lengths in centimeters. Thus $r = \dfrac{C}{2\pi} = \dfrac{20}{2\pi} = \dfrac{10}{\pi}$ and $dr = \dfrac{dC}{2\pi} = \dfrac{1}{2\pi}$. Now compute the differential dV and plug in the data.

$$\begin{aligned} V &= \tfrac{4}{3}\pi r^3 \\ dV &= 4\pi r^2\, dr \\ dV &= 4\pi\left(\frac{10}{\pi}\right)^2\left(\frac{1}{2\pi}\right) = \frac{200}{\pi^2}\ \text{cm}^3 \end{aligned}$$

19. (a) Isolate x step-by-step. The answer checks out.

$$\begin{aligned} 10\left(1 + e^{-x}\right)^{-1} &= 3 \\ \left(1 + e^{-x}\right)^{-1} &= \tfrac{3}{10} \\ 1 + e^{-x} &= \tfrac{10}{3} \\ e^{-x} &= \tfrac{7}{3} \\ -x &= \ln 7 - \ln 3 \\ x &= \ln 3 - \ln 7 = \ln \tfrac{3}{7} \end{aligned}$$

(b) Same drill. Check your answers!

$$\begin{aligned} \log_2(2x+1) &= 2 - \log_2(4x) \\ \log_2(2x+1) + \log_2(4x) &= 2 \\ \log_2\left((2x+1)(4x)\right) &= 2 \\ 8x^2 + 4x &= 2^2 = 4 \\ 8x^2 + 4x - 4 &= 0 \\ 2x^2 + x - 1 &= 0 \\ (2x-1)(x+1) &= 0 \\ x &= \tfrac{1}{2}, -1 \end{aligned}$$

Toss out $x = -1$. Only $x = \frac{1}{2}$ satisfies the *original* equation!

20. First note that when $t = 2$, we have $x = 6 + 4t = 14$, $y = 7 + 2t + \frac{1}{2}t^2 = 13$, $\frac{dx}{dt} = 4$ and $\frac{dy}{dt} = 2 + t = 4$. Apply the Pythagorean Theorem, then proceed.

$$\begin{aligned} z^2 &= x^2 + y^2 \\ 2z\frac{dz}{dt} &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \\ \frac{dz}{dt} &= \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{z} \\ \frac{dz}{dt} &= \frac{(14)(4) + (13)(4)}{\sqrt{14^2 + 13^2}} = \frac{108}{\sqrt{365}}\ \text{cm/s} \end{aligned}$$

Happy Halloween!