

Fall 2007 Math 151

Exam 3B: Solutions

Thu, 29/Nov

©2007 Art Belmonte

1. (d) Combine this indeterminate difference to obtain an indeterminate quotient, then use L'Hospital's Rule twice!

$$\lim_{x \rightarrow 1^+} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right) = \lim_{x \rightarrow 1^+} \frac{x \ln x + 1 - x}{(x-1) \ln x}$$

$$\stackrel{\text{L'H}}{=} \lim_{x \rightarrow 1^+} \frac{\ln x + 1 - 1}{\ln x + \frac{x-1}{x}} = \lim_{x \rightarrow 1^+} \frac{\ln x}{\ln x + 1 - x^{-1}}$$

$$\stackrel{\text{L'H}}{=} \lim_{x \rightarrow 1^+} \frac{\frac{1}{x}}{\frac{1}{x} + x^{-2}} = \frac{1}{2}$$

2. (b) Let y be the population (in billions) t years past 1990. The law of exponential growth gives $y = y_0 e^{kt} = 5e^{kt}$. Thus

$$6 = 5e^{10k}$$

$$k = \frac{1}{10} \ln \frac{6}{5}$$

$$y = 5e^{(\frac{1}{10} \ln \frac{6}{5})t} = 5 \left(\frac{6}{5} \right)^{t/10}$$

$$y(20) = 5 \left(\frac{6}{5} \right)^2 = \frac{36}{5} = 7.2 \text{ (billion).}$$

3. (d) Since $\frac{d}{dx} (e^{2x} \ln x) = 2e^{2x} \ln x + e^{2x} \left(\frac{1}{x} \right)$, we conclude that $e^{2x} \ln x$ is an antiderivative of $\left(\frac{1}{x} + 2 \ln x \right) e^{2x}$.

4. (e) There is a local maximum of f at $x = 5$ since the sign of f' changes from $+$ to $-$ there.

5. (b) The critical numbers of $f(x) = x^{2/3}(x-1)^2$ are values in the domain of f for which $f'(x)$ is 0 or undefined.

$$f'(x) = \frac{2}{3}x^{-1/3}(x-1)^2 + 2x^{2/3}(x-1)$$

$$f'(x) = x^{-1/3}(x-1) \left(\frac{2}{3}(x-1) + 2x \right), x \neq 0$$

$$f'(x) = x^{-1/3}(x-1) \left(\frac{8}{3}x - \frac{2}{3} \right), x \neq 0$$

Thus $f'(x) = 0$ when $x-1 = 0$ or $\frac{8}{3}x - \frac{2}{3} = 0$. This occurs for $x = \frac{1}{4}, 1$. We also suspect that $f'(0)$ is undefined. This is true since (for example) the right-hand limit of the difference quotient is ∞ , as we now show.

$$\lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^+} \frac{x^{2/3}(x-1)^2 - 0}{x - 0}$$

$$= \lim_{x \rightarrow 0^+} \frac{(x-1)^2}{x^{1/3}} = \frac{1}{0^+} = \infty$$

Hence the three critical numbers of f are $x = 0, \frac{1}{4}, 1$.

6. (d) Use the Closed Interval Method. First find the critical numbers of f that are in the interval of interest.

$$f(x) = x^3 - 6x^2 + 9x + 2 \text{ on } [-1, 2]$$

$$f'(x) = 3x^2 - 12x + 9 = 0$$

$$0 = x^2 - 4x + 3 = (x-1)(x-3)$$

$$x = 1, 3; \text{ only } 1 \text{ is in } [-1, 2]$$

Now compute values of f at this critical number and the endpoints of the interval.

x	-1	1	2
$f(x)$	-14	6	4

We conclude that the absolute minimum of f on $[-1, 2]$ is $f(-1) = -14$.

7. (c) Recall the diagram, *q.v.* With $x = 6$ ft and $\frac{dx}{dt} = 2$ ft/s, we see that

$$\theta = \sin^{-1}(x/10)$$

$$\frac{d\theta}{dt} = \frac{1}{\sqrt{1 - (x/10)^2}} \frac{dx}{dt} \frac{1}{10}$$

$$d\theta/dt = \frac{1/5}{4/5} = \frac{1}{4} \text{ rad/s.}$$

8. (e) Now $f(x) = \ln(x^2 e^{x^3}) = 2 \ln x + x^3$. So $f'(x) = \frac{2}{x} + 3x^2$ and thus $f'(1) = 5$. The point-slope formula gives $y - 1 = 5(x - 1)$ or $y = 5x - 4$.

9. (a) Recall $y = (\tan^{-1} x)^{\ln x}$. Use logarithmic differentiation.

$$\ln y = (\ln x) \ln(\tan^{-1} x)$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{x} \ln(\tan^{-1} x) + \ln x \frac{1}{\tan^{-1} x} \frac{1}{1+x^2}$$

$$y' = \frac{dy}{dx} = (\tan^{-1} x)^{\ln x} \left(\frac{\ln(\tan^{-1} x)}{x} + \frac{\ln x}{(1+x^2)\tan^{-1} x} \right)$$

$$y'(1) = \left(\frac{\pi}{4} \right)^0 (\ln \frac{\pi}{4} + 0) = \ln(\pi/4)$$

10. (c) Write $f(x) = \ln(\sec^2 x) = 2 \ln(\sec x)$. Then $f'(x) = 2 \frac{1}{\sec x} \sec x \tan x = 2 \tan x$.

11. (e) Now $\Delta x = \frac{b-a}{n} = \frac{6-0}{3} = 2$. With $f(x) = \frac{x-1}{x+1}$ on $[0, 6]$, the Midpoint Rule yields

$$\int_0^6 \frac{x-1}{x+1} dx \approx \sum_{k=1}^3 f(x_k) \Delta x$$

$$= 2(f(1) + f(3) + f(5))$$

$$= 2 \left(0 + \frac{2}{4} + \frac{4}{6} \right)$$

$$= 2 \left(\frac{1}{2} + \frac{2}{3} \right) = 1 + \frac{4}{3} = \frac{7}{3}$$

12. (c) We need to find the absolute maximum value of $g = f'$ on $[0, \pi]$. We'll use the Closed Interval Method. With $f(x) = x + \cos 2x$, we have $g(x) = f'(x) = 1 - 2 \sin 2x$.

- Let's find where $g'(x) = f''(x) = -4 \cos 2x = 0$ or $\cos 2x = 0$ on $[0, \pi]$. This implies $2x = \frac{\pi}{2}, \frac{3\pi}{2}$, whence $x = \frac{\pi}{4}, \frac{3\pi}{4} \in [0, \pi]$.
- Now compute values of $g(x) = f'(x) = 1 - 2 \sin 2x$ at these critical numbers and the endpoints of the interval.

x	0	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	π
$f'(x)$	1	-1	3	1

We conclude that the absolute maximum of f' on $[0, \pi]$ occurs at $x = \frac{3\pi}{4}$, where $f'(\frac{3\pi}{4}) = 3$.

13. (d) Recall that $f(x) = x^3 + x - 1$ on $[1, 2]$. By the Mean Value Theorem, there is a number $c \in (0, 2)$ such that $f'(c) = \frac{f(2) - f(0)}{2 - 0}$. Therefore,

$$3c^2 + 1 = \frac{9 - (-1)}{2} = 5$$

$$c = \pm \sqrt{\frac{4}{3}} = \pm 2/\sqrt{3}.$$

Hence $c = 2/\sqrt{3} \in (0, 2)$.

14. (b) Now $\mathbf{v}'(t) = \mathbf{a}(t) = \langle 4t, 6t \rangle$ implies $\mathbf{v}(t) = \langle 2t^2, 3t^2 \rangle + \mathbf{C}$, whence $\mathbf{v}(0) = \langle 1, -1 \rangle$ yields $\mathbf{C} = \langle 1, -1 \rangle$. Repeat: $\mathbf{r}'(t) = \mathbf{v}(t) = \langle 2t^2 + 1, 3t^2 - 1 \rangle$ implies $\mathbf{r}(t) = \langle \frac{2}{3}t^3 + t, t^3 - t \rangle + \mathbf{K}$. So $\mathbf{r}(0) = \langle 1, 0 \rangle$ gives $\mathbf{K} = \langle 1, 0 \rangle$. Thus $\mathbf{r}(t) = \langle \frac{2}{3}t^3 + t + 1, t^3 - t \rangle$. Hence $\mathbf{r}(3) = \langle 22, 24 \rangle$.

15. Let h be the height of the box and x its length and width. The volume of the box is $V = hx^2$. Its surface area (no top!) is $S = x^2 + 4hx = 1200$ whence $h = (1200 - x^2)/(4x)$, with the constraint that $0 < x < \sqrt{1200} = 20\sqrt{3}$. Thus

$$V = hx^2 = \frac{1200 - x^2}{4x} x^2 = 300x - \frac{1}{4}x^3$$

$$V' = 300 - \frac{3}{4}x^2$$

$$V'' = -\frac{3}{2}x$$

Now $V' = 0$ implies $x = \pm 20$; so $x = 20 \in (0, 20\sqrt{3})$.

Note that $V'' < 0$ for $0 < x < 20\sqrt{3}$. Accordingly the largest volume is $V(20) = 6000 - 2000 = 4000 \text{ cm}^3$.

16. The limit $\lim_{x \rightarrow 0} \left(\cos\left(\frac{x}{3}\right) \right)^{-9/x^2}$ involves an indeterminate power $1^{-\infty}$. Accordingly, let $y = \left(\cos\left(\frac{x}{3}\right) \right)^{-9/x^2}$. Then

$$\lim_{x \rightarrow 0} \ln y = \lim_{x \rightarrow 0} \frac{-9 \ln\left(\cos\left(\frac{x}{3}\right)\right)}{x^2}$$

$$\stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0} \frac{-9 \frac{-\frac{1}{3} \sin(x/3)}{\cos(x/3)}}{2x}$$

$$= \lim_{x \rightarrow 0} \frac{3 \tan(x/3)}{2x}$$

$$\stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0} \frac{\sec^2(x/3)}{2} = \frac{1}{2}.$$

Thus $\lim_{x \rightarrow 0} y = \lim_{x \rightarrow 0} e^{\ln y} = e^{1/2}$ or \sqrt{e} .

17. Use signed area! Now $\int_0^7 g(x) dx = \frac{1}{2}bh = \frac{1}{2}(1)(1) = \frac{1}{2}$.

- (a) $\int_0^2 g(x) dx = \frac{1}{2}bh = \frac{1}{2}(2)(4) = 4$;
 (b) $\int_2^6 g(x) dx = -\frac{1}{2}\pi r^2 = -\frac{1}{2}\pi(2)^2 = -2\pi$;
 (c) $\int_0^7 g(x) dx = \int_0^2 g + \int_2^6 g + \int_6^7 g = \frac{9}{2} - 2\pi$.

18. Antidifferentiate acceleration to obtain velocity and velocity to get position. Then resolve constants using numerical data.

$$\begin{aligned} v'(t) = a(t) &= 10 \sin t + 3 \cos t \\ x'(t) = v(t) &= -10 \cos t + 3 \sin t + C \\ x(t) &= -10 \sin t - 3 \cos t + Ct + K \\ 0 = x(0) &= K - 3 \implies K = 3 \\ 12 = x(2\pi) &= -3 + 2\pi C + 3 \implies C = 6/\pi \\ x(t) &= 3 + \frac{6t}{\pi} - 10 \sin t - 3 \cos t \end{aligned}$$

19. Recall that $f(x) = \sqrt{\tan x} = (\tan x)^{1/2}$ on $(0, \frac{\pi}{2})$.

- (a) Now $f'(x) = \frac{1}{2}(\tan x)^{-1/2} \sec^2 x > 0$ on $(0, \frac{\pi}{2})$, whence f is increasing on $(0, \frac{\pi}{2})$.
 (b) Thus $m = f(\frac{\pi}{4}) = 1$ and $M = f(\frac{\pi}{3}) = \sqrt[4]{3}$ are the least and greatest values of f on $[\frac{\pi}{4}, \frac{\pi}{3}]$, respectively.
 (c) Geometrically, $\int_a^b k dx = k(b-a)$ gives the area of a rectangle. So $\int_{\pi/4}^{\pi/3} m dx = \frac{\pi}{12}$ & $\int_{\pi/4}^{\pi/3} M dx = \frac{\pi}{12} \sqrt[4]{3}$. The average of these definite integrals gives a good estimate of $\int_{\pi/4}^{\pi/3} \sqrt{\tan x} dx$; namely, $\frac{\pi}{24} \left(1 + \sqrt[4]{3} \right)$.

20. Inscribe a rectangle with corners $(\pm x, \pm y)$ in the unit circle $x^2 + y^2 = 1$. Its area is $A = 4xy = 4x\sqrt{1-x^2}$, with $0 < x < 1$, where (x, y) is in the first quadrant. Now $A'(x) = 4\sqrt{1-x^2} - \frac{4x^2}{\sqrt{1-x^2}} = 0$ implies $x = \pm 1/\sqrt{2}$.

We toss out $-1/\sqrt{2} \notin (0, 1)$; keep $1/\sqrt{2} \in (0, 1)$. Notice that $A'(x) > 0$ for $0 < x < 1/\sqrt{2}$ and $A'(x) < 0$ for $1/\sqrt{2} < x < 1$. Therefore, we may conclude by the First Derivative Test for Absolute Extrema that the maximum area is $A(1/\sqrt{2}) = 2 \text{ cm}^2$.

Good luck on ALL your finals. Have a safe and enjoyable semester break!

— Art Belmonte, Course Coordinator