Math 151 Spring 2005 Exam II Solutions

1. D: Let $f(x) = \dfrac{1}{\sqrt{e^x}}$. By properties of exponents, $f(x) = \dfrac{1}{e^{x/2}} = e^{-x/2}$. By the chain rule, $f'(x) = -\dfrac{1}{2}e^{-x/2} = -\dfrac{1}{2\sqrt{e^x}}$.

2. A: We will use the chain rule to find $H'(x)$: $H(x) = f(g(x)) = f(3x^2 - 1)$. Thus $H'(x) = f'(3x^2 - 1)(6x) \Rightarrow H'(1) = f'(2)(6) = 5(6) = 30$.

3. C: By the product and chain rule, $\dfrac{d}{dx}(x \sin^2 x) = 1 * \sin^2 x + x * 2 \sin x \cos x$ $= \sin^2 x + 2x \sin x \cos x$.

4. B: $\lim_{x\to\infty} 3^{-x+1} = \lim_{x\to\infty} \dfrac{3}{3^x} = 0$.

5. D: Let $f(x) = \cos x$ and $a = \dfrac{\pi}{2}$. Then by the definition of the derivative,

$$f'(\frac{\pi}{2}) = \lim_{x\to\frac{\pi}{2}} \frac{f(x) - f(\frac{\pi}{2})}{x - \frac{\pi}{2}}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\cos x - \cos\frac{\pi}{2}}{x - \frac{\pi}{2}}$$

$= \lim_{x\to\frac{\pi}{2}} \dfrac{\cos x}{x - \frac{\pi}{2}}$. Now,

$$f'(\frac{\pi}{2}) = -\sin\frac{\pi}{2} = -1.$$

6. D: Use properties of logarithms: $\log_3 x + \log_3 x^2 = 6 \Rightarrow \log_3 x^3 = 6 \Rightarrow 3^6 = x^3 \Rightarrow x = 9$.

7. Graph missing on posted common exam.

8. C: The only False statement is C. $f(-1) = L(-1) = -1$.

9. E: Let $f^{-1}(3) = a$. Then $f(a) = 3 \Rightarrow 2e^{3a} - 1 = 3 \Rightarrow 2e^{3a} = 4$ $\Rightarrow e^{3a} = 2 \Rightarrow 3a = \ln 2$, giving $a = \dfrac{\ln 2}{3}$.

10. E: $L(x) = f(64) + f'(64)(x - 64)$. $f(64) = 4$ and $f'(64) = \dfrac{1}{48}$, thus $L(x) = 4 + \dfrac{1}{48}(x - 64)$. Now, $\sqrt[3]{60} \approx L(60) = 4 + \dfrac{1}{48}(-4) = \dfrac{47}{12}$.

11. C: $y = e^{ax}$, $y' = ae^{ax}$, $y'' = a^2e^{ax}$. Plugging these three functions into the given differential equation yields $a^2e^{ax} - 4ae^{ax} + 4e^{ax} = 0 \Rightarrow e^{ax}(a^2 - 4a + 4) = 0 \Rightarrow e^{ax}(a - 2)^2 = 0$. The only solution to this equation is $a = 2$ since $e^{ax} \neq 0$.

12. E: Newton's Method gives $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$. Since $x_1 = 1$, $x_2 = 1 - \dfrac{f(1)}{f'(1)}$, where $f(x) = x^3 + x - 3$. Now, $f(1) = -1$ and $f'(1) = 4$. thus $x_2 = 1 + \dfrac{1}{4} = \dfrac{5}{4}$.

13. B: We will differentiate $xy = 12$ with respect to t: $\dfrac{dx}{dt}y + \dfrac{dy}{dt}x = 0$. We are given $\dfrac{dx}{dt} = 4$, and we also know when $x = 4$, $y = 3$. Plugging these known values into our derivative gives $4 * 3 + \dfrac{dy}{dt} * 4 = 0$, giving $\dfrac{dy}{dt} = -3$.

14. a.) We will find the slope of the tangent by evaluating y' at the point $(-1, 1)$. We must differentiate implicitly.
$3y^2y' + 2xy + x^2y' = 3y'$
$\Rightarrow y'(3y^2 + x^2 - 3) = -2xy \Rightarrow y' = \dfrac{-2xy}{3y^2 + x^2 - 3}$.
Plugging in the point $(-1, 1)$, we get $y'(-1, 1) = 2$. Thus $m = 2$, so an equation of the tangent line is $y - 1 = 2(x + 1)$.

    b.) Using $y' = \dfrac{-2xy}{3y^2 + x^2 - 3}$, we will find y'' by the quotient rule:
$$y'' = \frac{(-2y - 2xy')(3y^2 + x^2 - 3) - (-2xy)(6yy' + 2x)}{(3y^2 + x^2 - 3)^2}.$$
Now we know $y'(-1, 1) = 2$ from part a.). So we will replace x with -1, y with 1 and y' with 2. This gives $y'' = -18$.

15. a.) Using the Quotient Rule:

$$y' = \frac{\cos x(\sin x + \cos x) - \sin x(\cos x - \sin x)}{(\sin x + \cos x)^2}$$
$$= \frac{\cos x \sin x + \cos^2 x - \sin x \cos x + \sin^2 x}{\sin^2 x + 2\sin x \cos x + \cos^2 x}$$
$$= \frac{1}{(\sin x + \cos x)^2}$$

b.) Using the Chain Rule:

$$y' = 2\sec(\sqrt{x})\sec(\sqrt{x})\tan(\sqrt{x}) * 1/2x^{-1/2}$$
$$= \frac{\sec^2(\sqrt{x})\tan(\sqrt{x})}{\sqrt{x}}$$

16. a.) $g'(1) = \dfrac{1}{f'(g(1))}$ Since $f(2) = 1$, $g(1) = 2$ since $g = f^{-1}$. Thus

$g'(1) = \dfrac{1}{f'(2)}$. Now by the Quotient Rule, $f'(x) = \dfrac{5}{(2x+1)^2}$, hence $f'(2) = \dfrac{5}{25} = \dfrac{1}{5}$. This gives $g'(1) = 5$.

b.) Let's first find $g(x) = f^{-1}(x)$.

Let $y = \dfrac{3x-1}{2x+1}$. Interchange x and y:

$x = \dfrac{3y-1}{2y+1}$. Now solve for y:

$x(2y+1) = 3y - 1 \Rightarrow y(3-2x) = x + 1$, giving $y = \dfrac{x+1}{3-2x}$. Therefore $g(x) = \dfrac{x+1}{3-2x}$, so by the Quotient Rule,

$g'(x) = \dfrac{5}{(3-2x)^2}$, and therefore $g'(1) = 5$. This agrees with our solution to part a.).

17. a.) $\mathbf{v}(t) = \mathbf{r}'(t) = \langle -6\sin(2t), 4\cos(2t)\rangle$. So the velocity at $t = \dfrac{\pi}{6}$ is

$$\mathbf{v}(\frac{\pi}{6}) = \left\langle -6\sin(\frac{\pi}{3}), 4\cos(\frac{\pi}{3})\right\rangle = \left\langle -3\sqrt{3}, 2\right\rangle.$$

b.) The speed at $t = \dfrac{\pi}{6}$ is

$$\left|\left\langle -3\sqrt{3}, 2\right\rangle\right| = \sqrt{(-3\sqrt{3})^2 + 2^2} = \sqrt{31}$$

c.) The acceleration is

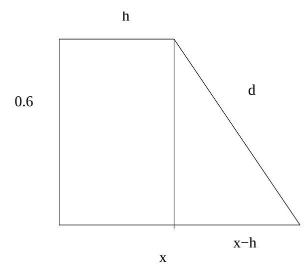
$\mathbf{a}(t) = \mathbf{v}'(t) = \langle -12\cos(2t), -8\sin(2t)\rangle$. So the acceleration at $t = \dfrac{\pi}{6}$ is

$$\mathbf{a}(\frac{\pi}{6}) = \left\langle -12\cos(\frac{\pi}{3}), -8\sin(\frac{\pi}{3})\right\rangle = \left\langle -6, -4\sqrt{3}\right\rangle.$$

d.) The slope if the tangent at $t = \dfrac{\pi}{6}$ is

$$\frac{dy/dt}{dx/dt} = \frac{2}{-3\sqrt{3}}.$$

18. a.) Let h = the horizontal distance the helicopter has travelled t hours after clocking the car with it's radar gun, x = horizontal distance along the ground at time t from the point directly below the helicopter at time $t = 0$ to the current position of the car, and d = distance from the helicopter to the the car. We know $\dfrac{dh}{dt} = 100$ and $\dfrac{dd}{dt} = -20$ when $d = 1$. We wish to find $\dfrac{dx}{dt}$.



b.) By Pythagorean Theorem, $d^2 = (x-h)^2 + (0.6)^2$. Differentiate with respect to t gives

$2d\dfrac{dd}{dt} = 2(x-h)(\dfrac{dx}{dt} - \dfrac{dh}{dt})$.

Now, when $d = 1$, we get $x - h = 0.8$. Plug all of this known information into our derivative yields

$2(1)(-20) = 2(0.8)(\dfrac{dx}{dt} - 100)$, thus $\dfrac{dx}{dt} = 75$ mph.