

Spring 2008
Math 151
Common Exam IA
Solution Set

1. With $f(x) = -x^3 - 5x + 3$, we see that

$$f(-2) = 21, \quad f(-1) = 9, \quad f(0) = 3, \quad f(1) = -3, \quad f(2) = -15,$$

and $f(3) = -39$. $f(x)$ changes sign in the interval $[0,1]$, so by the Intermediate Value Theorem, the equation $f(x) = 0$ has a solution in that interval—Choice (c). We know that there is no solution anywhere else because $f(x)$ is strictly decreasing.

2. $\lim_{x \rightarrow 0} \frac{1}{x} \sin\left(\frac{1}{x}\right)$ does not exist—Choice (e)—because if we set $h = \frac{1}{x}$, then $\lim_{x \rightarrow 0^+} \frac{1}{x} \sin\left(\frac{1}{x}\right) = \lim_{h \rightarrow \infty} h \sin h$. The amplitude of the oscillation is growing instead of decaying as we take the limit.

3. $f(x) = (x^2 - x - 1)(x^3 + x^2 + 1) \Rightarrow$

$$f'(x) = (x^2 - x - 1) \frac{d}{dx}(x^3 + x^2 + 1) + (x^3 + x^2 + 1) \frac{d}{dx}(x^2 - x - 1)$$

by the product rule for differentiation. Thus

$$\begin{aligned} f'(x) &= (x^2 - x - 1)(3x^2 + 2x) + (x^3 + x^2 + 1)(2x - 1) \\ &= (3x^4 - x^3 - 5x^2 - 2x) + (2x^4 + x^3 - x^2 + 2x - 1) \\ &= 5x^4 - 6x^2 - 1, \end{aligned}$$

which is Choice (b).

4. If $y = \sqrt{x}$, then $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$, and so $\left.\frac{dy}{dx}\right|_{x=9} = 1/6$. Therefore, the slope of the line tangent to the curve at $(9,3)$ is $\frac{1}{6}$. The equation of the tangent line is $y - 3 = \frac{1}{6}(x - 9)$, which we may write as

$$y = \frac{1}{6}x + \frac{3}{2}. \quad \text{(Choice (d))}$$

5. Choose a convenient point on the line $y = 5x - 9$, say $Q = (2, 1)$. The point of interest is $P = (-1, 3)$, and the distance formula is

$$d = \frac{|\overrightarrow{PQ} \cdot \vec{a}|}{|\vec{a}|}, \quad \vec{a} \perp \{y = 5x - 9\}.$$

If we write the equation of the line in the form $5x - y = 9$, we see that $\langle 5, -1 \rangle$ is such an \vec{a} . With

$$\overrightarrow{PQ} = \overrightarrow{(-1, 3)(2, 1)} = \langle 3, -2 \rangle,$$

it follows that

$$d = \frac{|\langle 3, -2 \rangle \cdot \langle 5, -1 \rangle|}{|\langle 5, -1 \rangle|} = \frac{17}{\sqrt{26}}. \quad \text{(Choice (a))}$$

6. The function $f(x)$ is increasing in the regions $-1 < x < 1$ and $x > 2$. This means that the graph of $f'(x)$ is above the x -axis in those regions. This rules out Choices (a), (c), and (e). On the other hand, $f(x)$ is decreasing in the regions $x < -1$ and $1 < x < 2$, so the graph of $f'(x)$ is below the x -axis in those regions. This rules out Choice (d), and Choice (b) satisfies the conditions.

$$7. \vec{v}(t) = \vec{r}'(t) = \left\langle \frac{d}{dt}(50t), \frac{d}{dt}(120t - 16t^2) \right\rangle \\ = \langle 50, 120 - 32t \rangle,$$

so $\vec{v}(3) = \langle 50, 24 \rangle$. (Choice (d))

$$8. \text{proj}_{\vec{b}} \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \vec{b}$$

With $\vec{a} = \langle -10, 7 \rangle$ and $\vec{b} = \langle 5, 12 \rangle$,

$\vec{a} \cdot \vec{b} = 34$ and $|\vec{b}| = 13$, so

$$\text{proj}_{\vec{b}} \vec{a} = \frac{34}{169} \langle 5, 12 \rangle \quad (\text{Choice (b)})$$

$$9. y = \frac{2x^2 - 3x + 1}{x^2 - 1} = \frac{(2x - 1)(x - 1)}{(x + 1)(x - 1)} = \frac{2x - 1}{x + 1}, \text{ so the only vertical asymptote is } x = -1.$$

(Choice (a))

$$10. \text{(a) } \lim_{x \rightarrow 3^+} f(x) = 1 \text{ and } \lim_{x \rightarrow 3^-} f(x) = -1.$$

$$\text{(b) } \lim_{x \rightarrow 3^-} f(x) = 6 \text{ and } f(3) = 9.$$

$$\text{(c) } \lim_{x \rightarrow 3^-} f(x) = 0 \text{ and } f(3) = 0 \text{ and } \lim_{x \rightarrow 3^+} f(x) = 0.$$

$$\text{(d) } \lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} \frac{1}{(x - 3)^2} = \infty.$$

(e) $f(3)$ is undefined.

\therefore Choice (c) is the function that is continuous at $x = 3$.

Show your work. Partial credit is possible.
No credit will be given to unsupported answers.

11. An automobile safety advertisement claims that hitting a stone wall at 60 mph is equivalent to having your vehicle roll off the roof of a building of height h . Assuming that the acceleration due to gravity is 32 ft./sec^2 , find h . (Recall from your driving test that $60 \text{ mph} = 88 \text{ ft./sec.}$) (8 points)

Note. An introduction to anti-derivatives is supposedly given sometime during the 4th week of classes—at the request of the physics department—so this question is (supposedly) fair game. However, most instructors simply do not have the time to introduce anti-derivatives that early, so this problem was graded as a bonus question.

If we regard “up” as the positive direction, then the acceleration due to gravity is just the constant -32 (in units of feet and seconds). Since acceleration is the derivative of velocity $v(t)$ as a function of time t , it follows that

$$v(t) = -32t + c \quad (1)$$

for some constant c . If the vehicle begins to fall at time $t = 0$, then $v(0)$ is zero, but this means that $c = 0$, so we have

$$v(t) = -32t. \quad (2)$$

On the other hand, vertical velocity is the derivative of altitude $y(t)$ as a function of time t , and therefore

$$y(t) = -16t^2 + C \quad (3)$$

for some constant C . If zero altitude is street-level, then $y(0) = h$, which means that $C = h$, and so

$$y(t) = h - 16t^2. \quad (4)$$

If $v(t) = -88$ and $y(t) = 0$ both hold, then $(2) \Rightarrow t = \frac{11}{4}$ and therefore $(4) \Rightarrow h = 121$.

12. Find the derivative $f'(x)$ of each function $f(x)$. (6 points each)

(a) $f(x) = \frac{x^2 + 3x + 2}{x^2 - 1}$

Solution. $f(x) = \frac{(x+1)(x+2)}{(x+1)(x-1)} = \frac{x+2}{x-1}$, so by the quotient rule,

$$f'(x) = \frac{1(x-1) - 1(x+2)}{(x-1)^2} = -\frac{3}{(x-1)^2}.$$

If one does not see the initial reduction, one may still use the quotient rule:

$$f'(x) = \frac{(2x+3)(x^2-1) - 2x(x^2+3x+2)}{(x^2-1)^2}.$$

This answer is acceptable, but notice that

$$f'(x) = \frac{-3x^2 - 6x - 3}{(x^2 - 1)^2} = -3 \frac{(x^2 + 2x + 1)}{(x-1)^2(x+1)^2} = -\frac{3}{(x-1)^2}.$$

(b) $f(x) = x^2 + 3x\sqrt{x} - 4x + 9\sqrt{x} - 8 + \frac{2}{\sqrt{x}} + \frac{5}{x}$

Solution.

$$\begin{aligned} f(x) &= x^2 + 3x^{3/2} - 4x + 9x^{1/2} - 8 + 2x^{-1/2} + 5x^{-1} \\ f'(x) &= 2x + \frac{9}{2}x^{1/2} - 4 + \frac{9}{2}x^{-1/2} - x^{-3/2} - 5x^{-2}. \end{aligned}$$

This answer is acceptable, but we may also write

$$f'(x) = 2x + \frac{9}{2}\sqrt{x} - 4 + \frac{9}{2\sqrt{x}} - \frac{1}{x\sqrt{x}} - \frac{5}{x^2}.$$

13. Let $f(x) = \sqrt{x+2}$ for $x \geq -2$. Calculate $f'(\frac{1}{4})$ by using the definition of derivative. (10 points)

Solution.

$$\begin{aligned} f' \left(\frac{1}{4} \right) &= \lim_{x \rightarrow \frac{1}{4}} \frac{f(x) - f(\frac{1}{4})}{x - \frac{1}{4}} \\ &= \lim_{x \rightarrow \frac{1}{4}} \frac{\sqrt{x+2} - \frac{3}{2}}{x - \frac{1}{4}} = \lim_{x \rightarrow \frac{1}{4}} \frac{(\sqrt{x+2} - \frac{3}{2})(\sqrt{x+2} + \frac{3}{2})}{(x - \frac{1}{4})(\sqrt{x+2} + \frac{3}{2})} \\ &= \lim_{x \rightarrow \frac{1}{4}} \frac{(x+2) - \frac{9}{4}}{(x - \frac{1}{4})(\sqrt{x+2} + \frac{3}{2})} = \lim_{x \rightarrow \frac{1}{4}} \frac{x - \frac{1}{4}}{(x - \frac{1}{4})(\sqrt{x+2} + \frac{3}{2})} \\ &= \lim_{x \rightarrow \frac{1}{4}} \frac{1}{\sqrt{x+2} + \frac{3}{2}} = \frac{1}{\frac{3}{2} + \frac{3}{2}} = \frac{1}{3} \end{aligned}$$

14. Find the Cartesian equation of the line perpendicular to the line given by the parametric equations

$$x = 4s + 3$$

$$y = 3s - 4$$

and passing through the point (2,1). (8 points)

Solution. One approach is to note that $s = \frac{x-3}{4}$ and therefore

$$y = 3 \left(\frac{x-3}{4} \right) - 4 = \frac{3}{4}x - \frac{25}{4}.$$

This means that the slope of the given line is $\frac{3}{4}$ and so the slope of any perpendicular line is $-\frac{4}{3}$.

Thus, the equation of the desired line is $y - 1 = -\frac{4}{3}(x - 2)$.

Another approach is to note that $\langle 4, 3 \rangle$ has the direction of the given line and is therefore perpendicular to the desired line. If (x, y) is an arbitrary point on the desired line, this means that $\langle 4, 3 \rangle \perp \overline{(2, 1)(x, y)} = \langle x - 2, y - 1 \rangle$. Thus

$$\langle 4, 3 \rangle \cdot \langle x - 2, y - 1 \rangle = 0,$$

and so

$$4(x - 2) + 3(y - 1) = 0.$$

15. Calculate the following limits. You may **not** use L'Hospital's Rule. (4 points each)

(a) $\lim_{x \rightarrow \infty} \frac{1 - \cos x}{x^2}$

Solution. Since $-1 \leq \cos x \leq 1$, we know that $0 \leq 1 - \cos x \leq 2$, and so

$$0 \leq \frac{1 - \cos x}{x^2} \leq \frac{2}{x^2}.$$

On the other hand, $\lim_{x \rightarrow \infty} \frac{2}{x^2} = 0$, so it follows from the Squeeze Theorem that $\lim_{x \rightarrow \infty} \frac{1 - \cos x}{x^2} = 0$.

(b) $\lim_{x \rightarrow 1} \frac{\sqrt{x} - 1}{x^2 - 1}$

Solution.

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{\sqrt{x} - 1}{x^2 - 1} &= \lim_{x \rightarrow 1} \frac{(\sqrt{x} - 1)(\sqrt{x} + 1)}{(x^2 - 1)(\sqrt{x} + 1)} \\ &= \lim_{x \rightarrow 1} \frac{x - 1}{(x^2 - 1)(\sqrt{x} + 1)} = \lim_{x \rightarrow 1} \frac{x - 1}{(x - 1)(x + 1)(\sqrt{x} + 1)} \\ &= \lim_{x \rightarrow 1} \frac{1}{(x + 1)(\sqrt{x} + 1)} = \frac{1}{(1 + 1)(1 + 1)} = \frac{1}{4}. \end{aligned}$$

(c) $\lim_{x \rightarrow \infty} \frac{3x^3 - x + 5}{5x^3 + x^2 - 3}$

Solution.

$$\lim_{x \rightarrow \infty} \frac{3x^3 - x + 5}{5x^3 + x^2 - 3} = \lim_{x \rightarrow \infty} \frac{3 - \frac{1}{x^2} + \frac{5}{x^3}}{5 + \frac{1}{x} - \frac{3}{x^3}}.$$

Since $\lim_{x \rightarrow \infty} \frac{1}{x^n} = 0$ for every positive integer n , it follows that

$$\lim_{x \rightarrow \infty} \frac{3x^3 - x + 5}{5x^3 + x^2 - 3} = \frac{3 - 0 + 5(0)}{5 + 0 - 3(0)} = \frac{3}{5}.$$