

**Spring 2005 Math 152**  
**Exam 2B: Solutions**  
**Mon, 28/Mar**      **©2005, Art Belmonte**

For specificity, lengths are in centimeters unless stated otherwise.

1. (c)

- For non-*STEPS* folks: We have

$$\bar{x} = \frac{\sum_{k=1}^3 m_k x_k}{\sum_{k=1}^3 m_k} = \frac{(3)(1) + (5)(-2) + (7)(3)}{3 + 5 + 7} = \frac{14}{15}.$$

(Note that this only gives you the  $x$ -coordinate of the center of mass, not the entire center of mass.)

- For *STEPS* folks: Let  $\mathbf{p} = [3, 5, 7]$  be the row vector of masses and

$$\mathbf{r} = \begin{bmatrix} 1 & 2 \\ -2 & 5 \\ 3 & 1 \end{bmatrix}$$

be a matrix whose rows are position vectors of the points. Then  $m = \text{sum}(\mathbf{p}) = 3 + 5 + 7 = 15$  is the total mass.

The center of mass is

$$\begin{aligned} [\bar{x}, \bar{y}] &= \frac{1}{m} (\mathbf{p}\mathbf{r}) \\ &= \frac{1}{15} [3 - 10 + 21, 6 + 25 + 7] \\ &= \left[ \frac{14}{15}, \frac{38}{15} \right]. \end{aligned}$$

So the  $x$ -coordinate of the center of mass is  $\bar{x} = \frac{14}{15}$ .  
*(Remark:  $\mathbf{p}\mathbf{r}$  represents matrix multiplication, realized by taking the dot products of rows with columns. This immediately extends to 3-D center of mass problems in Calc 3.)*

2. (c) Let  $f(x) = \ln x$ . We'll determine  $K = \max_{2 \leq x \leq 5} |f''(x)|$ , then employ the Trapezoidal error estimate.

- Now  $f'(x) = \frac{1}{x} = x^{-1}$  and  $f''(x) = -x^{-2}$ . Thus

$$K = \max_{2 \leq x \leq 5} \left( \frac{1}{x^2} \right) = \frac{1}{4}.$$

- Therefore,

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} = \frac{\frac{1}{4}(5-2)^3}{12(4)^2} = \frac{27}{3(4)^4} = \frac{9}{256}.$$

3. (d) The improper integral converges to  $\pi/4$ .

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^2+1} dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^2+1} dx \\ &= \lim_{t \rightarrow \infty} \left( \tan^{-1} x \right) \Big|_1^t \\ &= \lim_{t \rightarrow \infty} \left( \tan^{-1} t - \tan^{-1} 1 \right) \\ &= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \end{aligned}$$

4. (a) The step size is  $h = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4}$ . Partition points are  $\left\{ 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2 \right\}$ . The Trapezoidal Rule gives

$$T_n = \text{step size} \times (\text{avg of endpt func vals} + \sum \text{interior func vals})$$

$$\begin{aligned} \text{or } T_4 &= \frac{1}{2}h \left( f(1) + 2f\left(\frac{5}{4}\right) + 2f\left(\frac{3}{2}\right) + 2f\left(\frac{7}{4}\right) + f(2) \right) \\ &= \frac{1}{8} \left( \ln 1 + 2 \ln \frac{5}{4} + 2 \ln \frac{3}{2} + 2 \ln \frac{7}{4} + \ln 2 \right) \\ &= \frac{1}{8} \left( 2 \ln \frac{5}{4} + 2 \ln \frac{3}{2} + 2 \ln \frac{7}{4} + \ln 2 \right) \end{aligned}$$

5. (b) The slope of the curve at a point  $(x, y)$  is  $x^3 y^2$  and the curve passes through  $(0, \frac{1}{2})$ . This gives us a separable differential equation  $dy/dx = x^3 y^2$  and an initial condition  $y(0) = \frac{1}{2}$ . Solve this initial value problem.

$$\begin{aligned} y^{-2} dy &= x^3 dx \\ -\frac{1}{y} &= -y^{-1} = \frac{1}{4}x^4 + C \\ -2 &= C \quad [\text{Substitute the initial condition.}] \\ -\frac{1}{y} &= \frac{1}{4}x^4 - 2 \\ y &= \frac{1}{2 - \frac{1}{4}x^4} = \frac{4}{8 - x^4} \end{aligned}$$

6. (e) We have

$$\begin{aligned} 0 &\leq \int_0^{\infty} \frac{1}{\sqrt{x} + e^{10x}} dx \leq \int_0^{\infty} \frac{1}{e^{10x}} dx = \int_0^{\infty} e^{-10x} dx \\ &= \lim_{t \rightarrow \infty} \int_0^t e^{-10x} dx \\ &= \lim_{t \rightarrow \infty} \left( -\frac{1}{10} e^{-10x} \right) \Big|_0^t \\ &= \lim_{t \rightarrow \infty} \left( -\frac{1}{10} e^{-10t} + \frac{1}{10} \right) = \frac{1}{10}. \end{aligned}$$

Therefore,  $\int_0^{\infty} \frac{1}{\sqrt{x} + e^{10x}} dx$  converges to a value  $L \leq \frac{1}{10}$ .

7. (c) Put the differential equation in standard linear form.

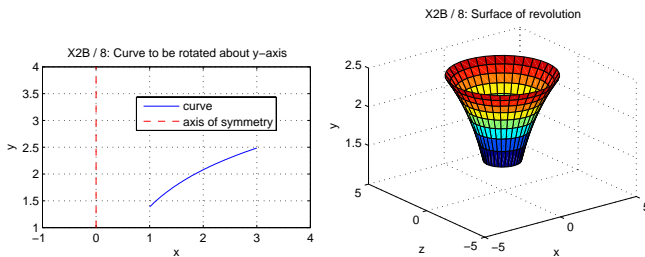
$$\frac{dy}{dx} + \frac{4x}{x^2+1} y = \frac{x^3}{x^2+1}$$

Then an integrating factor is

$$\begin{aligned} \mu &= \exp\left(\int \frac{4x}{x^2+1} dx\right) = \exp(2 \ln(x^2+1)) \\ &= \exp(\ln(x^2+1)^2) \\ &= (x^2+1)^2. \end{aligned}$$

8. (e) The surface area is given by

$$S = \int 2\pi r ds = \int_a^b 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_1^3 2\pi x \sqrt{1 + \left(\frac{1}{x}\right)^2} dx.$$



9. (a) The arc length of the curve

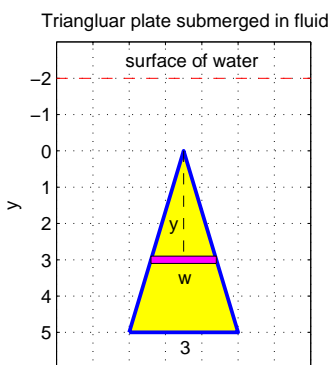
$$\mathbf{r}(t) = [x(t), y(t)] = [3t + 1, 4 - t], \quad 1 \leq t \leq 3,$$

$$\begin{aligned} \text{is } L &= \int_a^b \|\mathbf{r}'(t)\| dt = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2} dt \\ &= \int_1^3 \sqrt{(3)^2 + (-1)^2} dt = \int_1^3 \sqrt{10} dt = 2\sqrt{10} \approx 6.32 \text{ cm.} \end{aligned}$$

10. (d) With  $y = \frac{2}{3}(x-1)^{3/2}$ ,  $1 \leq x \leq 4$ , the arc length is

$$\begin{aligned} L &= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\ &= \int_1^4 \sqrt{1 + (\sqrt{x-1})^2} dx \\ &= \int_1^4 x^{1/2} dx \\ &= \left. \frac{2}{3}x^{3/2} \right|_1^4 = \frac{16}{3} - \frac{2}{3} = \frac{14}{3}. \end{aligned}$$

11. Here is a diagram of the submerged plate.



- Via similar triangles we have  $3/5 = w/y$ ; so  $w = \frac{3}{5}y$ .
- A slice of the plate is at variable depth  $z$ . Build up your integral step-by-step:  $P$ ,  $dA$ ,  $dF$ ,  $F = \int dF$ .

$$P = \delta z = \delta(y - (-2)) = \delta(y + 2)$$

$$dA = w dy = \frac{3}{5}y dy$$

$$dF = P dA = \frac{3}{5}\delta y(y + 2) dy$$

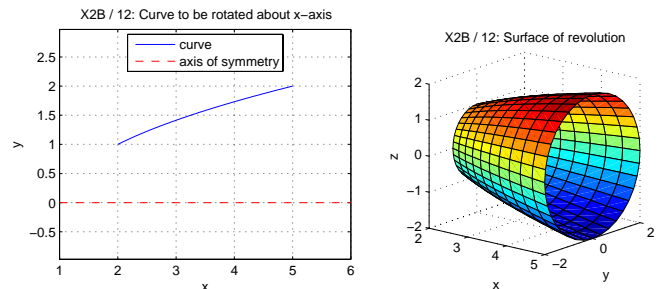
$$F = \frac{3}{5}\delta \int_0^5 y^2 + 2y dy$$

$$= \frac{3}{5}\delta \left(\frac{1}{3}y^3 + y^2\right) \Big|_0^5$$

$$= \frac{3}{5}(60) \left(\frac{125}{3} + 25\right) = 2400 \text{ lb.}$$

12. The surface area is  $S = \int 2\pi r ds$  or

$$\begin{aligned} \int_c^d 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy &= \int_1^2 2\pi y \sqrt{1 + (2y)^2} dy \\ &= \pi \int_1^2 (1 + 4y^2)^{1/2} 2y dy \\ &= \frac{\pi}{4} \left(\frac{2}{3}\right) (1 + 4y^2)^{3/2} \Big|_1^2 \\ &= \frac{\pi}{6} (17\sqrt{17} - 5\sqrt{5}) \approx 30.85 \text{ cm}^2. \end{aligned}$$



13. The density  $\rho$  is constant. First compute the mass

$$m = \iint_D \rho dA = \rho \iint_D dA = \rho A = \rho \left(\frac{1}{4}\pi (2)^2\right) = \rho\pi$$

then compute  $\bar{x}$ , the  $x$ -coordinate of the centroid.

$$\bar{x} = \frac{1}{m} \iint_D \rho x dA \quad \text{[STEPS folks here; others below.]}$$

$$= \frac{1}{\rho\pi} \int_0^2 \int_0^{\sqrt{4-x^2}} \rho x dy dx$$

$$= \frac{1}{\pi} \int_0^2 xy \Big|_{y=0}^{y=\sqrt{4-x^2}} dx \quad [\rho \text{ is constant}]$$

$$\bar{x} = \frac{1}{\pi} \int_0^2 (4-x^2)^{1/2} x dx \quad \text{[i.e., } \frac{1}{A} \int_a^b xf(x) dx]$$

$$= \left(-\frac{1}{2}\right) \left(\frac{1}{\pi}\right) \left(\frac{2}{3}\right) (4-x^2)^{3/2} \Big|_0^2$$

$$= 0 - \left(-\frac{8}{3\pi}\right) = \frac{8}{3\pi}$$

14. (a) Factor to see that the differential equation is separable.

$$\begin{aligned}\frac{dy}{dt} &= 1 + t - yt - y = (t + 1) - y(t + 1) \\ \frac{dy}{dt} &= (t + 1)(1 - y) = -(t + 1)(y - 1) \\ \frac{1}{y - 1} dy &= -(t + 1) dt \\ \ln|y - 1| &= -\frac{1}{2}(t + 1)^2 + A \\ |y - 1| &= e^{A - (t+1)^2/2} = e^A e^{-(t+1)^2/2} \\ \pm(y - 1) &= B e^{-(t+1)^2/2} \\ y &= 1 + C e^{-(t+1)^2/2}\end{aligned}$$

- (b) The differential equation is already in standard linear form (SLF):  $y' - 2y = x$ . An integrating factor is  $\mu = \exp\left(\int -2 dx\right) = e^{-2x}$ . Multiply the SLF by  $\mu$  and solve.

$$\begin{aligned}e^{-2x} y' - 2e^{-2x} y &= x e^{-2x} \\ (e^{-2x} y)' &= x e^{-2x} \\ e^{-2x} y &= \left(-\frac{1}{2}x - \frac{1}{4}\right) e^{-2x} + C \\ y &= -\frac{1}{2}x - \frac{1}{4} + C e^{2x} \\ 1 = y(0) &= -\frac{1}{4} + C \quad \text{[initial condition]} \\ C &= \frac{5}{4} \\ y &= -\frac{1}{2}x - \frac{1}{4} + \frac{5}{4} e^{2x}\end{aligned}$$

In the third step we integrated by parts. With  $u = x$  and  $dv = e^{-2x} dx$  we have  $du = dx$  and  $v = -\frac{1}{2}e^{-2x}$ . Therefore,

$$\begin{aligned}\int x e^{-2x} dx &= -\frac{1}{2}x e^{-2x} + \int \frac{1}{2} e^{-2x} dx \\ &= \left(-\frac{1}{2}x - \frac{1}{4}\right) e^{-2x} + C\end{aligned}$$

15. Let  $y = y(t)$  be the amount of salt in the tank at time  $t$ . Since the tank initially contains pure water, we have  $y(0) = 0$  kg of salt in the tank at the start. The classical balance law gives

$$\begin{aligned}\frac{dy}{dt} &= \text{rate in} - \text{rate out} \\ \frac{dy}{dt} &= \left(10 \frac{\text{L}}{\text{min}} \times \frac{1}{2} \frac{\text{kg}}{\text{L}}\right) - \left(10 \frac{\text{L}}{\text{min}} \times \frac{y \text{ kg}}{1000 \text{ L}}\right) \\ \frac{dy}{dt} &= 5 - \frac{1}{100}y \quad \frac{\text{kg}}{\text{min}} \\ y' + \frac{1}{100}y &= 5.\end{aligned}$$

The differential equation is both separable and linear. Let's find an integrating factor for the standard linear form (SLF) in the last equation above:  $\mu = \exp\left(\int \frac{1}{100} dt\right) = e^{t/100}$ .

Multiply the SLF by  $\mu$  and solve.

$$\begin{aligned}e^{t/100} y' + \frac{1}{100} e^{t/100} y &= 5e^{t/100} \\ (e^{t/100} y)' &= 5e^{t/100} \\ e^{t/100} y &= 500e^{t/100} + C \\ y &= 500 + C e^{-t/100} \\ 0 = y(0) &= 500 + C \quad \text{[initial condition]} \\ C &= -500 \\ y &= 500 - 500e^{-t/100} \\ y(10) &= 500 - 500e^{-10/100} \approx 47.58 \text{ kg}\end{aligned}$$