Spring 2007 Math 152
Exam 1A: Problems and Solutions
Mon, 19/Feb ©2007, Art Belmonte

1. (e) Compute $\int_0^2 \frac{5x+7}{x^2+4x+3}\,dx$.

   - Split the rational integrand into a sum of partial fractions.

$$\begin{aligned} \frac{5x+7}{(x+1)(x+3)} &= \frac{A}{x+1}+\frac{B}{x+3} \\ 5x+7 &= A(x+3)+B(x+1) \\ 5x+7 &= (A+B)x+(3A+B) \end{aligned}$$

   - Equate coefficients of like terms.

$$\begin{aligned} A+B &= 5 \\ 3A+B &= 7 \end{aligned}$$

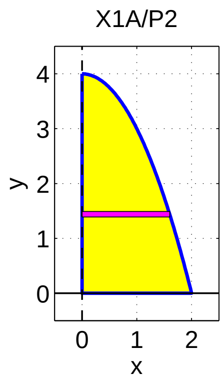
   Subtracting the second equation from the first gives $-2A=-2$ or $A=1$. Thus $B=5-A=4$.

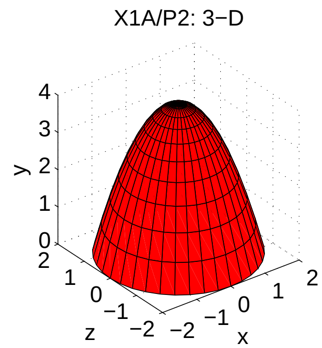
   - Integrate term-by-term.

$$\begin{aligned} &\int_0^2 \frac{1}{x+1}+\frac{4}{x+3}\,dx \\ =\ & \left(\ln|x+1|+4\ln|x+3|\right)\Big|_0^2 \\ =\ & (\ln 3+4\ln 5)-(\ln 1+4\ln 3) \\ =\ & 4\ln 5-3\ln 3 \end{aligned}$$

   Here we have recalled that $\ln 1=0$.

2. (a) Find the volume of the solid generated when the region in the first quadrant bounded by $x=0$, $y=0$, and $y=4-x^2$ is revolved about the y-axis.





   - Via disks, the volume is

$$\begin{aligned} V &= \int \pi r^2\,dy = \int \pi x^2\,dy \\ &= \pi\int_0^4 4-y\,dy \\ &= \left(-\frac{\pi}{2}(4-y)^2\right)\Big|_0^4 \\ &= 0-(-8\pi)=8\pi. \end{aligned}$$

3. (b) Find the value of $\int_0^2 x^3e^{x^2}\,dx$.

   - Use integration by parts. Compute an antiderivative, then apply the FTC. Let $u=x^2$ and $dv=e^{x^2}x\,dx$. Then $du=2x\,dx$ and $v=\frac{1}{2}e^{x^2}$. Hence $\int x^3e^{x^2}\,dx=\frac{1}{2}x^2e^{x^2}-\int e^{x^2}x\,dx=\frac{1}{2}\left(x^2-1\right)e^{x^2}$.

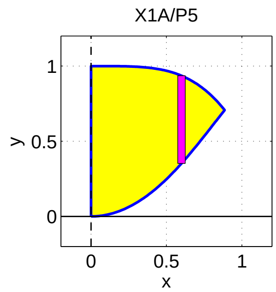
   - Thus $\int_0^2 x^3e^{x^2}\,dx=\frac{1}{2}\left(x^2-1\right)e^{x^2}\Big|_0^2=\frac{3}{2}e^4-\left(-\frac{1}{2}\right)$ or $\frac{3e^4+1}{2}$.

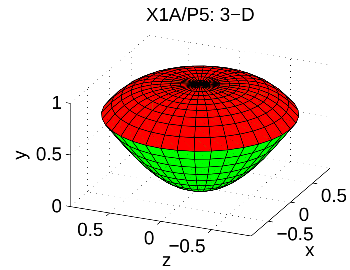
4. (c) Find the average value of $f(x)=\frac{x}{\sqrt{x^2+16}}$ on the interval [0, 3].

   - The average value is given by

$$\begin{aligned} f_{ave} &= \frac{1}{b-a}\int_a^b f(x)\,dx \\ &= \frac{1}{3-0}\int_0^3 \left(x^2+16\right)^{-1/2}x\,dx \\ &= \left(\tfrac{1}{3}\right)\left(\tfrac{1}{2}\right)(2)\left(x^2+16\right)^{1/2}\Big|_0^3 \\ &= \frac{5}{3}-\frac{4}{3}=\frac{1}{3}. \end{aligned}$$

5. (d) The region in the first quadrant bounded by $y=\sin\left(x^2\right)$ and $y=\cos\left(x^2\right)$, $0\le x\le\frac{1}{2}\sqrt{\pi}$, is revolved about the y-axis. Find the volume of the resulting solid.

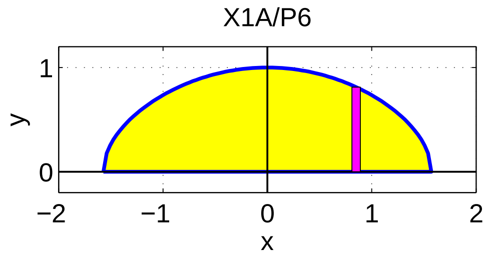


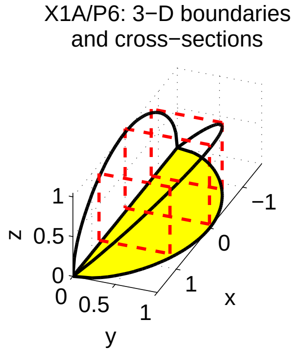


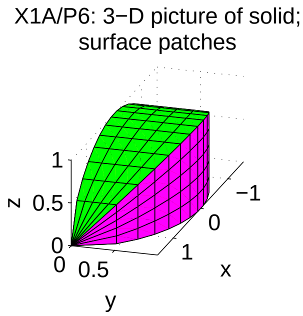
   - Via cylindrical shells, the volume swept out by the revolving the stated region about the y-axis is given by

$$\begin{aligned} V &= \int 2\pi r h\,dx \\ &= \pi\int_0^{\sqrt{\pi}/2} 2x\left(\cos\left(x^2\right)-\sin\left(x^2\right)\right)dx \\ &= \pi\left(\sin\left(x^2\right)+\cos\left(x^2\right)\right)\Big|_0^{\sqrt{\pi}/2} \\ &= \sqrt{2}\,\pi-\pi \end{aligned}$$

6. (e) The base of a solid is bounded by $y=\sqrt{\cos x}$, $-\frac{1}{2}\pi\le x\le\frac{1}{2}\pi$, and the x-axis. Each cross-section perpendicular to the x-axis is a square region whose bottom is sitting on this base. Find the volume of the solid.







- The volume by slicing is

$$\begin{aligned} V &= \int y^2\,dx \\ &= \int_{-\pi/2}^{\pi/2} \cos x\,dx \\ &= \sin x\Big|_{-\pi/2}^{\pi/2} \\ &= 1-(-1) = 2. \end{aligned}$$

7. (c) A 10-lb object hangs over a ledge at the end of a 20-ft chain that weighs $\frac{1}{2}$ lb per foot. Find the total work (in ft-lb) done hauling the object up to the ledge.

   - The work done lifting the object itself is

   $$W_{object} = (10)(20) = 200 \text{ ft-lb.}$$

   - The work done lifting the rope is

   $$W_{rope} = \int_0^{20} \tfrac{1}{2}x\,dx = \tfrac{1}{4}x^2\Big|_0^{20} = 100 \text{ ft-lb.}$$

   - The total amount of work is $200 + 100 = 300$ ft-lb.

8. (d) Compute $\displaystyle\int_1^e \frac{\ln x}{x^2}\,dx$.

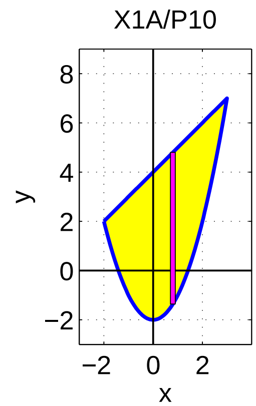
   - Use integration by parts. Compute an antiderivative, then apply the FTC. Let $u = \ln x$ and $dv = x^{-2}\,dx$. Then $du = \frac{1}{x}\,dx$ and $v = -x^{-1}$. Accordingly, $\displaystyle\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} + \int x^{-2}\,dx = -\frac{1+\ln x}{x}$.
   - Thus $\displaystyle\int_1^e \frac{\ln x}{x^2}\,dx = \left(-\frac{1+\ln x}{x}\right)\Big|_1^e = -\frac{2}{e} - (-1)$ or $1 - \dfrac{2}{e}$.

9. (b) For a certain type of linear spring, the force required to keep it stretched 2 feet beyond its natural length is 5 lb. How much work (in ft-lb) is done stretching this spring 4 feet beyond its natural length?

   - Via Hooke's Law, we have $F(x) = kx$ or $5 = 2k$, whence $k = \frac{5}{2}$.
   - The work done is $W = \int_a^b F(x)\,dx$ or

   $$\int_0^4 \tfrac{5}{2}x\,dx = \tfrac{5}{4}x^2\Big|_0^4 = 20 - 0 = 20 \text{ ft-lb.}$$

10. (d) Find the area of the region bounded by the line $y = x + 4$ and the parabola $y = x^2 - 2$.



   - When the curves intersect, their y-coordinates are equal. This yields

   $$\begin{aligned} x+4 &= x^2-2 \\ 0 &= x^2-x-6 \\ 0 &= (x+2)(x-3) \\ x &= -2, 3. \end{aligned}$$

   - At $x = 0$, an interior point of $[-2, 3]$, the line's y-coordinate is 4, whereas that of the parabola is -2. Thus the line lies above the parabola.
   - Hence the area is given by

   $$\begin{aligned} A &= \int_{-2}^{3} (x+4) - \left(x^2-2\right)\,dx \\ &= \int_{-2}^{3} 6 + x - x^2\,dx \\ &= \left(6x + \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right)\Big|_{-2}^{3} \\ &= \left(18 + \tfrac{9}{2} - 9\right) - \left(-12 + 2 + \tfrac{8}{3}\right) \\ &= \tfrac{27}{2} - \left(-\tfrac{22}{3}\right) \\ &= \frac{81+44}{6} = \frac{125}{6}. \end{aligned}$$

11. Evaluate $\displaystyle\int \frac{x+1}{\left(x^2+4\right)^{3/2}}\,dx$.

   - Use trigonometric substitution. Let $x = 2\tan\theta$. Then $dx = 2\sec^2\theta\,d\theta$.
   - Changing variables, the integral becomes

   $$\begin{aligned} &\int \frac{2\tan\theta + 1}{8\sec^3\theta}\cdot 2\sec^2\theta\,d\theta \\ =& \int \tfrac{1}{2}\sin\theta + \tfrac{1}{4}\cos\theta\,d\theta \\ =& -\tfrac{1}{2}\cos\theta + \tfrac{1}{4}\sin\theta + C. \end{aligned}$$

[continued...]

- Finally, we express the antiderivative in terms of the original variable x.

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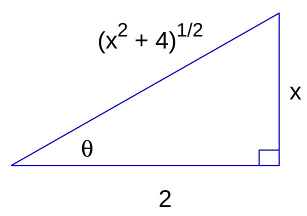
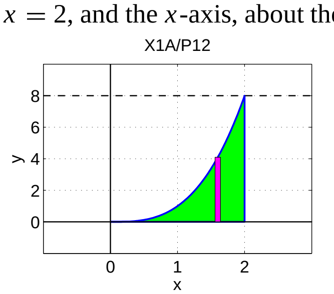


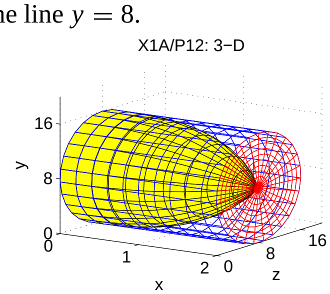
$$-\frac{1}{2}\frac{2}{\sqrt{x^2+4}}+\frac{1}{4}\frac{x}{\sqrt{x^2+4}}+C$$

$$=\frac{x}{4\sqrt{x^2+4}}-\frac{1}{\sqrt{x^2+4}}+C$$

$$\text{or}\quad \frac{x-4}{4\sqrt{x^2+4}}+C$$

12. Find the volume of the solid generated by revolving the region in the first quadrant bounded by $y = x^3$, the line $x = 2$, and the x-axis, about the line $y = 8$.





- Via washers, the volume is

$$\begin{aligned}
V &= \int \pi r_o^2 - \pi r_i^2\,dx \\
&= \pi\int_0^2 8^2-\left(8-x^3\right)^2\,dx \\
&= \pi\int_0^2 16x^3-x^6\,dx \\
&= \pi\left(4x^4-\tfrac{1}{7}x^7\right)\Big|_0^2 \\
&= \left(64-\tfrac{128}{7}\right)\pi-0=\tfrac{320}{7}\pi
\end{aligned}$$

13. Find $\int \cos^3 3\theta \sin^{-2} 3\theta\,d\theta$.

- Use the trigonometric identity $\sin^2 x + \cos^2 x = 1$.

$$\begin{aligned}
&\int \cos 3\theta\left(1-\sin^2 3\theta\right)\sin^{-2}3\theta\,d\theta \\
=\;& \int (\sin 3\theta)^{-2}\cos 3\theta-\cos 3\theta\,d\theta \\
=\;& -\tfrac{1}{3}(\sin 3\theta)^{-1}-\tfrac{1}{3}\sin 3\theta+C \\
\text{or}\;& -\tfrac{1}{3}(\csc 3\theta+\sin 3\theta)+C
\end{aligned}$$

14. Compute $\int \cos t \cos 4t\,dt$.

- Use a trigonometric product formula

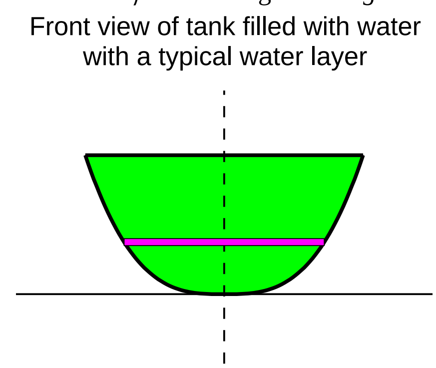
$$\cos A\cos B=\tfrac{1}{2}\big(\cos(A-B)+\cos(A+B)\big)$$

plus the fact the the cosine function is even: $\cos(-\theta)=\cos\theta$.

- Integrate the transformed integral as follows.

$$\begin{aligned}
\int \cos t\cos 4t\,dt &= \int \tfrac{1}{2}\left(\cos(-3t)+\cos 5t\right)\,dt \\
&= \tfrac{1}{2}\int \cos 3t+\cos 5t\,dt \\
&= \tfrac{1}{2}\left(\tfrac{1}{3}\sin 3t+\tfrac{1}{5}\sin 5t\right)+C \\
\text{or}\quad & \tfrac{1}{6}\sin 3t+\tfrac{1}{10}\sin 5t+C
\end{aligned}$$

15. A tank full of water has the depicted shape of a solid of revolution obtained by rotating about the y-axis the region in the first quadrant bounded by $x = (4y)^{1/3}$ and the lines $y = 2$ and $x = 0$. Find the work (in joules) required to pump the water out of the tank. Lengths are in meters. The density of water is $\rho = 1000\ \text{kg/m}^3$ and $g = 9.8\ \text{m/s}^2$.



- From the diagram on the exam depicting a typical layer of water, we follow the "march of the differentials" for volume, mass, force, and work.

$$\begin{aligned}
dV &= \pi r^2\,dy=\pi x^2\,dy=\pi\,(4y)^{2/3}\,dy \\
dm &= \rho\,dV=\rho\pi\,(4y)^{2/3}\,dy \\
dF &= (dm)\,g=g\rho\pi\,(4y)^{2/3}\,dy \\
dW &= (dF)\,(D)=g\rho\pi\,(4y)^{2/3}\,(2-y)\,dy
\end{aligned}$$

- Set up the and dispatch the work integral. (This begs for calculator firepower. Perhaps in future terms. . .)

$$\begin{aligned}
W &= \int dW=g\rho\pi\int_0^2 2\,(4)^{2/3}y^{2/3}-4^{2/3}y^{5/3}\,dy \\
&= g\rho\pi\left(2\,(4)^{2/3}\left(\tfrac{3}{5}\right)y^{5/3}-4^{2/3}\left(\tfrac{3}{8}\right)y^{8/3}\right)\Big|_0^2 \\
&= g\rho\pi\left(\tfrac{48}{5}-\tfrac{48}{8}\right)-0=48g\rho\pi\left(\tfrac{8-5}{40}\right) \\
&= \tfrac{18}{5}g\rho\pi=\tfrac{18}{5}\left(\tfrac{98}{10}\right)(1000)\,\pi=35{,}280\pi\ \text{joules}
\end{aligned}$$