Spring 2007 Math 152 Exam 2A Answers Mon, 26/Mar/2007

1. (a) $\int_0^\infty xe^{-x}\,dx = 1$ [improper integral; integration by parts]

2. (b) $\dfrac{dy}{dt} = 1 - \dfrac{y}{40}$, $y(0) = 35$ [classical balance law: net rate = rate-in − rate-out]

3. (b) Let $\mathbf{p} = \begin{bmatrix} 7 & 10 & 13 \end{bmatrix}$ be a row vector of the masses and $\mathbf{r} = \begin{bmatrix} -3 & 2 \\ 3 & 5 \\ 4 & 3 \end{bmatrix}$ a matrix whose rows are position vectors of the coordinates. Let $m = \operatorname{sum}(\mathbf{p}) = 30$. Then the position vector of the center of mass is given by $\mathrm{CM} = [\bar{x}, \bar{y}] = \dfrac{1}{m}\mathbf{pr} = [61/30, 103/30]$.

4. (e) The differential equation $y' + y\tan x = \sec x$ is in standard linear form (SLF). Multiply by the integrating factor $\mu = \sec x$, then antidifferentiate and isolate y to obtain $y = \sin x + K\cos x$.

5. (e) Given $y'/x = e^{x-y} = e^x/e^y$, separate variables then antidifferentiate and isolate y to obtain $y = \ln\left((x-1)\,e^x + C\right)$.

6. (e) The surface area is $\int_0^{\sqrt[4]{15}} 2\pi\left(x^3/3\right)\sqrt{1 + \left(x^2\right)^2}\,dx = 7\pi$.

7. (c) The arc length is $\int_{\sqrt{3}}^{\sqrt{24}} \sqrt{\left(t^2\right)^2 + (t)^2}\,dt = 39$.

8. (d) Note that $0 < \int_1^\infty \dfrac{e^{-x}}{x}\,dx \le \int_1^\infty e^{-x}\,dx = \dfrac{1}{e} \approx 0.37 < 1$. By a comparison theorem, we conclude that $\int_1^\infty \dfrac{e^{-x}}{x}\,dx$ converges to a value L between 0 and 1.

9. Now $f(x) = x^{-4/3}$ has an infinite discontinuity at $x = 0$. Accordingly, $\int_{-1}^8 x^{-4/3}\,dx = \int_{-1}^0 x^{-4/3}\,dx + \int_0^8 x^{-4/3}\,dx$. Since both $\int_{-1}^0 x^{-4/3}\,dx$ and $\int_0^8 x^{-4/3}\,dx$ diverge to ∞, we conclude that $\int_{-1}^8 x^{-4/3}\,dx$ diverges to ∞.

10. (a) The Trapezoidal Rule gives $T_4 = 1\left(\frac{1}{2}(1.0000 + 0.8776) + 0.9922 + 0.9689 + 0.9305\right) = 3.8304$.

    (b) Now $f''(x) = -\frac{1}{64}\cos(x/8)$, whence $K = \left|f''(0)\right| = \frac{1}{64}$. Thus $\dfrac{\frac{1}{64}(4-0)^3}{12n^2} < \dfrac{1}{10^8}$ implies $n > \sqrt{10^8/12} \approx 2886.75$; for example, choose $n = 2887$ or 3000. (The final inequality suffices.)

11. Let $\delta = k$ be the constant density of the region D. The mass of the region is $m = \iint_D \delta\,dA = \int_0^4 \int_0^{3x^2/2} k\,dy\,dx = 32k$. The position vector of the center of mass of the region is

$$\mathrm{CM} = [\bar{x}, \bar{y}] = \frac{1}{m}\iint_D \delta\,[x, y]\,dA = \frac{1}{32k}\int_0^4 \int_0^{3x^2/2} k\,[x, y]\,dy\,dx = \left[3, \frac{36}{5}\right] = [3, 7.2].$$

Naturally, students may compute the center of mass using moments as in Section 9.6.

12. The window is a vertical plate at variable depth. Build up the integral for hydrostatic force step by step.

$$\begin{aligned} P &= \rho g\,(\text{depth}) = \rho g\,(4 - y) \\ dA &= w\,dy = 2x\,dy = 2\,(4y)^{1/4}\,dy \\ dF &= P\,dA = 2\rho g\,(4y)^{1/4}\,(4 - y)\,dy \\ F &= \int_0^4 2\rho g\,(4y)^{1/4}\,(4 - y)\,dy \\ &= \frac{1024}{45}\rho g = \frac{2{,}007{,}040}{9} \approx 2.23 \times 10^5\ \mathrm{N} \end{aligned}$$

Spring 2007 Math 152
Exam 2: Executive Summary
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Chapters and Sections

8: Techniques of Integration

8.8 Approximate Integration

8.9 Improper Integrals

9: Further Applications of Integration

9.1 [Separable First-Order] Differential Equations

9.2 First-Order Linear Equations

9.3 Arc Length

9.4 Area of a Surface of Revolution

9.5 Moments and Centers of Mass

9.6 Hydrostatic Pressure and Force

Fundamental Concepts

Legend

$h = \Delta x = \frac{b-a}{n}$ $\quad dA = dx\,dy = dy\,dx$ $\quad K = \max_{a \le x \le b} \left| f''(x) \right|$

$x_k = a + kh;\, k = 0, \ldots, n$ $\quad z$: depth $\quad M = \max_{a \le x \le b} \left| f^{(4)}(x) \right|$

8.8

$$L_n = h \sum_{k=0}^{n-1} f(x_k) \quad R_n = h \sum_{k=1}^{n} f(x_k)$$

$$M_n = h \sum_{k=1}^{n} f\left(\frac{x_{k-1} + x_k}{2}\right)$$

$$T_n = h\left(\frac{f(x_0) + f(x_n)}{2} + \sum_{k=1}^{n-1} f(x_k)\right)$$

$$S_n = \frac{h}{3}\left(f(x_0) + f(x_n) + 4\sum_{i=0}^{n/2-1} f(x_{2i+1}) + 2\sum_{j=1}^{n/2-1} f(x_{2j})\right)$$

$$\left|E_{T_n}\right| \le \frac{K(b-a)^3}{12n^2} \qquad \left|E_{M_n}\right| \le \frac{K(b-a)^3}{24n^2}$$

$$\left|E_{S_n}\right| \le \frac{M(b-a)^5}{180n^4}$$ [For Simpson's Rule, n is even!]

8.9 Use *limits* and/or comparison theorems to compute improper integrals.

9.1 $dy/dx = f(x,y) \Longrightarrow g(y)\,dy = h(x)\,dx \Longrightarrow G(y) = H(x) + C \Longrightarrow y = G^{-1}(H(x) + C)$

9.2

$$y' + p(x)\,y = q(x)$$
$$\mu(x) = \exp\left(\int p(x)\,dx\right)$$
$$\mu(x)\,y' + \mu(x)\,p(x)\,y = \mu(x)\,q(x)$$
$$(\mu(x)\,y)' = \mu(x)\,q(x)$$
$$y = \frac{1}{\mu(x)}\left(\int \mu(x)\,q(x)\,dx + C\right)$$

9.3

$$L = \int ds \qquad ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

9.4 $S = \int 2\pi r\,ds$

9.5 Let $\mathbf{p} = \begin{bmatrix} m_1 & m_2 & \cdots & m_n \end{bmatrix}$, $m = \sum_{k=1}^{n} m_k$, and

$$\mathbf{r} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix}.$$ Then $\mathrm{CM} = [\bar{x}, \bar{y}] = \frac{1}{m}\,\mathbf{pr}$.

Let a planar region D have density δ. Then its mass is $m = \iint_D \delta\,dA$ and its center of mass is given by $\mathrm{CM} = [\bar{x}, \bar{y}] = \frac{1}{m}\iint_D \delta\,[x, y]\,dA$.

9.6 $P = \delta z = \rho g z,\ dA = w\,dy,\ dF = P\,dA,\ F = \int dF$

Additional Details

- $\displaystyle\int_a^\infty \frac{1}{x^p}\,dx$ converges for $p > 1$ and diverges for $p \le 1$.
- Comparison theorems for improper integrals: Let f and g be continuous on (a, ∞) with $f \ge g \ge 0$. [Just think of areas under curves above the x-axis and the following assertions are clear.]
  1. If $\int_a^\infty f(x)\,dx$ converges, then so does $\int_a^\infty g(x)\,dx$. Moreover, $0 \le \int_a^\infty g(x)\,dx \le \int_a^\infty f(x)\,dx = L$.
  2. If $\int_a^\infty g(x)\,dx$ diverges, then so does $\int_a^\infty f(x)\,dx$. Moreover, $\int_a^\infty g(x)\,dx = \int_a^\infty f(x)\,dx = \infty$.
- If a differential equation has an initial condition, $y(x_0) = y_0$, then this initial condition is used to resolve the constant involved in the general solution to the differential equation using algebra.

General framework for an [improper] integral

Let $(a, b) \subset \mathbb{R}$. Here a is either a real number or $-\infty$ and b is either a real number or ∞. Label $x_0 = a$ and $x_m = b$. Let $\{x_i\}_{i=1}^{m-1}$ be real numbers such that $x_0 < x_1 < \cdots < x_m$ and define $D = (a, b) \setminus \{x_i\}_{i=1}^{m-1}$. [In other words, D is (a, b) with the points $x_1, \ldots, x_{m-1}$ removed.] Suppose that $f : D \to \mathbb{R}$ is continuous and that f is discontinuous at $x_1, \ldots, x_{m-1}$. Finally, let $c_i \in (x_{i-1}, x_i)$, $i = 1, \ldots, m$. [Note that f is continuous at the interior points c_i.] Then we define

$$\int_a^b f(x)\,dx = \sum_{i=1}^{m}\left(\lim_{t \to x_{i-1}^+}\int_t^{c_i} f(x)\,dx + \lim_{t \to x_i^-}\int_{c_i}^{t} f(x)\,dx\right),$$

provided that *all* these limits exist.