

1. (b) Solve the differential equation $y'/x = e^{x-y}$.

- The differential equation $y'/x = e^{x-y} = e^x/e^y$ is separable.
- Thus $e^y dy = xe^x dx$ implies $e^y = (x - 1)e^x + C$, whence $y = \ln((x - 1)e^x + C)$.
- Note that we used integration by parts to compute $\int xe^x dx$. Indeed, let $u = x$ and $dv = e^x dx$. Then $du = dx$ and $v = e^x$. Therefore,

$$\int xe^x dx = xe^x - \int e^x dx = (x - 1)e^x.$$

2. (e) Does $\int_1^\infty \frac{e^{-x}}{x} dx$ converge?

- Observe that $0 < \frac{e^{-x}}{x} \leq e^{-x}$ on $[1, \infty)$.
- Moreover,

$$\begin{aligned} \int_1^\infty e^{-x} dx &= \lim_{t \rightarrow \infty} \int_1^t e^{-x} dx \\ &= \lim_{t \rightarrow \infty} (-e^{-x}) \Big|_1^t \\ &= \lim_{t \rightarrow \infty} \left(-e^{-t} + \frac{1}{e}\right) = \frac{1}{e} < 1. \end{aligned}$$

- By a comparison theorem, we conclude that $\int_1^\infty \frac{e^{-x}}{x} dx$ converges to a value L between 0 and 1.

3. (c) Find the arc length of the curve $x = t^3/3, y = t^2/2, \sqrt{3} \leq t \leq \sqrt{24}$.

- The arc length is $L = \int ds$.
- We have

$$\begin{aligned} \int ds &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_{\sqrt{3}}^{\sqrt{24}} \sqrt{(t^2)^2 + (t)^2} dt \\ &= \int_{\sqrt{3}}^{\sqrt{24}} \sqrt{t^4 + t^2} dt \\ &= \int_{\sqrt{3}}^{\sqrt{24}} (t^2 + 1)^{1/2} t dt \\ &= \frac{1}{2} \left(\frac{2}{3}\right) (t^2 + 1)^{3/2} \Big|_{\sqrt{3}}^{\sqrt{24}} \\ &= \frac{1}{3} (125) - \frac{1}{3} (8) = \frac{117}{3} = 39. \end{aligned}$$

4. (c) A tank contains 200 L of brine, initially with 35 kg of salt in solution. Brine containing 0.2 kg/L of salt enters the tank at the rate of 5 L/min. It flows out of the tank at the same rate. The mixture in the tank is kept uniform by constant stirring. If $y(t)$ is the amount of salt in the tank at time t , which of the following initial value problems models the behavior of $y(t)$?

- The classical balance law states that net rate = rate-in - rate-out.

$$\begin{aligned} \frac{dy}{dt} &= \left(0.2 \frac{\text{kg}}{\text{L}}\right) \left(5 \frac{\text{L}}{\text{min}}\right) - \left(\frac{y \text{ kg}}{200 \text{ L}}\right) \left(5 \frac{\text{L}}{\text{min}}\right) \\ \frac{dy}{dt} &= 1 - \frac{y}{40} \end{aligned}$$

- The initial condition is $y(0) = 35$.

5. (b) Find $\int_0^\infty xe^{-x} dx$.

- First compute the indefinite integral $\int xe^{-x} dx$ via integration by parts. Let $u = x$ and $dv = e^{-x} dx$. Then $du = dx$ and $v = -e^{-x}$. Accordingly, we have $\int xe^{-x} dx = -xe^{-x} + \int e^{-x} dx = -(x + 1)e^{-x}$.
- Therefore,

$$\begin{aligned} \int_0^\infty xe^{-x} dx &= \lim_{t \rightarrow \infty} \int_0^t xe^{-x} dx \\ &= \lim_{t \rightarrow \infty} (-(x + 1)e^{-x}) \Big|_0^t \\ &= \lim_{t \rightarrow \infty} \left(-\frac{t + 1}{e^t} + 1\right) = 1, \end{aligned}$$

$$\text{since } \lim_{t \rightarrow \infty} \frac{t + 1}{e^t} \stackrel{\text{L'H}}{=} \lim_{t \rightarrow \infty} \frac{1}{e^t} = 0.$$

6. (c) Solve the differential equation $y' + y \tan x = \sec x$.

- The differential equation $y' + y \tan x = \sec x$ is in standard linear form (SLF).
- An integrating factor is $\mu = \exp\left(\int \tan x dx\right) = \exp\left(\int \frac{\sin x}{\cos x} dx\right) = \exp(-\ln(\cos x)) = \sec x$.
- Multiplying the SLF by the integrating factor $\mu = \sec x$, we obtain $y' \sec x + y \sec x \tan x = \sec^2 x$ or $(y \sec x)' = \sec^2 x$.
- Thus $y \sec x = \tan x + K$, whence $y = \sin x + K \cos x$.

7. (e) Point masses of mass 7, 10, and 13 are positioned at $(-3, 2)$, $(3, 5)$, and $(4, 3)$, respectively. Find the center of mass of the system.

- Let $\mathbf{p} = \begin{bmatrix} 7 & 10 & 13 \end{bmatrix}$ be a row vector of the masses and $\mathbf{r} = \begin{bmatrix} -3 & 2 \\ 3 & 5 \\ 4 & 3 \end{bmatrix}$ a matrix whose rows are position vectors of the coordinates. Let m be the sum of the masses: $7 + 10 + 13 = 30$. [Continued on next page.]

- Then the position vector of the center of mass is given by $\text{CM} = [\bar{x}, \bar{y}] = \frac{1}{m} \mathbf{pr}$, as follows.

$$\begin{aligned} \frac{1}{m} \mathbf{pr} &= \frac{1}{30} \begin{bmatrix} 7 & 10 & 13 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 3 & 5 \\ 4 & 3 \end{bmatrix} \\ &= \frac{1}{30} [-21 + 30 + 52, 14 + 50 + 39] \\ &= [61/30, 103/30] \end{aligned}$$

8. (a) Find the area of the surface generated by rotating the Cartesian curve $y = x^3/3$, $0 \leq x \leq \sqrt[4]{15}$, about the x -axis.

- The surface area is given by $S = \int 2\pi r ds$.
- We have

$$\begin{aligned} \int 2\pi r ds &= \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\ &= \int_0^{\sqrt[4]{15}} 2\pi \left(x^3/3\right) \sqrt{1 + (x^2)^2} dx \\ &= \frac{2\pi}{3} \int_0^{\sqrt[4]{15}} (1 + x^4)^{1/2} x^3 dx \\ &= \frac{2\pi}{3} \frac{1}{4} \int_0^{\sqrt[4]{15}} (1 + x^4)^{1/2} 4x^3 dx \\ &= \frac{\pi}{6} \left(\frac{2}{3}\right) (1 + x^4)^{3/2} \Big|_0^{\sqrt[4]{15}} \\ &= \frac{\pi}{9} (64) - \frac{\pi}{9} (1) = \frac{63}{9} \pi = 7\pi. \end{aligned}$$

9. An aquarium has an odd-shaped window that opens to public view. The water level in the aquarium is level with the top of the window, which ranges from floor level at its middle to 4 meters high at the top. The window is bounded by the curve $y = \frac{1}{4}x^4$, $-2 \leq x \leq 2$, and the line $y = 4$, as shown below. Find the hydrostatic force pushing against the window. Water has a density of $\rho = 1000 \text{ kg/m}^3$; also, $g = 9.8 \text{ m/s}^2$.

- The window is a vertical plate at variable depth. Build up the integral for hydrostatic force step by step.

$$\begin{aligned} P &= \rho g (\text{depth}) = \rho g (4 - y) \\ dA &= w dy = 2x dy = 2(4y)^{1/4} dy \\ dF &= P dA = 2\rho g (4y)^{1/4} (4 - y) dy \\ F &= \int_0^4 2\rho g (4y)^{1/4} (4 - y) dy \\ &= 2\rho g (4)^{1/4} \int_0^4 4y^{1/4} - y^{5/4} dy \\ &= 2\rho g (4)^{1/4} \left(\frac{16}{5}y^{5/4} - \frac{4}{9}y^{9/4}\right) \Big|_0^4 \\ &= 2\rho g \left(4^{6/4} \left(\frac{16}{5}\right) - 4^{10/4} \left(\frac{4}{9}\right)\right) - 0 \\ &= 2 \left(\frac{2^7}{5} - \frac{2^7}{9}\right) \rho g = 2^8 \left(\frac{9-5}{45}\right) \rho g = \frac{2^{10}}{45} \rho g \\ &= \frac{1024}{45} (1000) \left(\frac{98}{10}\right) \\ &= \frac{2,007,040}{9} \approx 2.23 \times 10^5 \text{ N} \end{aligned}$$

10. Find the centroid (center of mass of a region of constant density) of the region in the first quadrant bounded by $y = \frac{3}{2}x^2$, the x -axis, and the line $x = 4$. [See **Notes** at end!]

- Let $\delta = k$ be the constant density of the region D . The mass of the region is $m = \iint_D \delta dA$, which we now compute.

$$\begin{aligned} m &= \iint_D \delta dA = \int_0^4 \int_0^{3x^2/2} k dy dx \\ &= \int_0^4 ky \Big|_{y=0}^{y=3x^2/2} dx \\ &= \int_0^4 \left(\frac{3kx^2}{2} - 0\right) dx \\ &= \frac{kx^3}{2} \Big|_0^4 = 32k - 0 = 32k \end{aligned}$$

- The position vector of the center of mass of the region is $\text{CM} = [\bar{x}, \bar{y}] = \frac{1}{m} \iint_D \delta [x, y] dA$. We have

$$\begin{aligned} &\frac{1}{m} \iint_D \delta [x, y] dA \\ &= \frac{1}{32k} \int_0^4 \int_0^{3x^2/2} k [x, y] dy dx \\ &= \frac{1}{32} \left[\int_0^4 \int_0^{3x^2/2} x dy dx, \int_0^4 \int_0^{3x^2/2} y dy dx \right] \\ &= \frac{1}{32} \left[\int_0^4 xy \Big|_{y=0}^{y=3x^2/2} dx, \int_0^4 \frac{1}{2}y^2 \Big|_{y=0}^{y=3x^2/2} dx \right] \\ &= \frac{1}{32} \left[\frac{3}{2} \int_0^4 x^3 dx, \frac{9}{8} \int_0^4 x^4 dx \right] \\ &= \frac{1}{32} \left[\frac{3}{8}x^4 \Big|_0^4, \frac{9}{40}x^5 \Big|_0^4 \right] \\ &= \frac{1}{32} \left[\frac{3(2)^8}{8}, \frac{9(2)^{10}}{40} \right] \\ &= \left[3, \frac{36}{5} \right] = [3, 7.2]. \end{aligned}$$

11. Consider the curve $y = f(x) = \cos(x/8)$ on the interval $0 \leq x \leq 4$.

- (a) Use the Trapezoidal Rule with 4 intervals to estimate

$\int_0^4 \cos(x/8) dx$. You may use the following table.

x	0	1	2	3	4
$\cos(x/8)$	1.0000	0.9922	0.9689	0.9305	0.8776

- The Trapezoidal Rule gives

$$\begin{aligned} T_4 &= 1 \left(\frac{1.0000 + 0.8776}{2} \right. \\ &\quad \left. + 0.9922 + 0.9689 + 0.9305 \right) \\ &= 3.8304. \end{aligned}$$

- (b) The difference between the actual value of an integral and the number obtained by using the Trapezoidal Rule with n intervals is at most

$$\frac{K(b-a)^3}{12n^2} \quad (*)$$

where K is the maximum of $|f''(x)|$ on the interval of integration. Determine a reasonable choice for n so that if n intervals were used instead of 4 as in part (a), the computed value of $\int_0^4 \cos(x/8) dx$ would be guaranteed by (*) to be within 10^{-8} of the actual value.

- Now $f''(x) = -\frac{1}{64} \cos(x/8)$, whence

$$K = |f''(0)| = \frac{1}{64}. \text{ Thus } \frac{\frac{1}{64}(4-0)^3}{12n^2} < \frac{1}{10^8}$$

implies $n > \sqrt{\frac{10^8}{12}} \approx 2886.75$. So choose

$n = 2887$. [The final inequality sufficed.]

12. Taking into account anything that might make the integral improper, find $\int_{-1}^8 x^{-4/3} dx$.

- Now $f(x) = x^{-4/3}$ has an infinite discontinuity at $x = 0$. Accordingly,

$$\int_{-1}^8 x^{-4/3} dx = \int_{-1}^0 x^{-4/3} dx + \int_0^8 x^{-4/3} dx.$$

Since both $\int_{-1}^0 x^{-4/3} dx$ and $\int_0^8 x^{-4/3} dx$ diverge to ∞ , we conclude that $\int_{-1}^8 x^{-4/3} dx$ diverges to ∞ ; details follow.

- We have

$$\begin{aligned} \int_{-1}^0 x^{-4/3} dx &= \lim_{t \rightarrow 0^-} \int_{-1}^t x^{-4/3} dx \\ &= \lim_{t \rightarrow 0^-} \left(-3x^{-1/3} \right) \Big|_{-1}^t \\ &= \lim_{t \rightarrow 0^-} \left(\frac{-3}{\sqrt[3]{t}} - 3 \right) = \infty. \end{aligned}$$

- Similarly,

$$\begin{aligned} \int_0^8 x^{-4/3} dx &= \lim_{t \rightarrow 0^+} \int_t^8 x^{-4/3} dx \\ &= \lim_{t \rightarrow 0^+} \left(-3x^{-1/3} \right) \Big|_t^8 \\ &= \lim_{t \rightarrow 0^+} \left(-\frac{3}{2} + \frac{3}{\sqrt[3]{t}} \right) = \infty. \end{aligned}$$

Notes

Here is another way to compute the center of mass in Problem 10.

- The mass is

$$\begin{aligned} m &= \int_a^b \delta(f(x) - g(x)) dx \\ &= \int_0^4 k \left(\frac{3}{2}x^2 - 0 \right) dx \\ &= \frac{1}{2}kx^3 \Big|_0^4 = 32k - 0 = 32k. \end{aligned}$$

- The moment with respect to the x -axis is

$$\begin{aligned} M_y &= \int_a^b \delta x (f(x) - g(x)) dx \\ &= \int_0^4 kx \left(\frac{3}{2}x^2 - 0 \right) dx \\ &= \int_0^4 \frac{3}{2}kx^3 dx \\ &= \frac{3}{8}kx^4 \Big|_0^4 = 96k - 0 = 96k. \end{aligned}$$

- The moment with respect to the y -axis is

$$\begin{aligned} M_x &= \int_a^b \delta \left(\frac{1}{2}f^2(x) - \frac{1}{2}g^2(x) \right) dx \\ &= \int_0^4 \frac{1}{2}k \left(\left(\frac{3}{2}x^2 \right)^2 - 0 \right) dx \\ &= \int_0^4 \frac{9}{8}kx^4 dx \\ &= \frac{9}{40}kx^5 \Big|_0^4 = \frac{9(2)^7}{5}k. \end{aligned}$$

- Accordingly, the center of mass is

$$\begin{aligned} \text{CM} = [\bar{x}, \bar{y}] &= \left[\frac{M_y}{m}, \frac{M_x}{m} \right] \\ &= \left[\frac{96k}{32k}, \frac{9(2)^7 k/5}{2^5 k} \right] \\ &= \left[3, \frac{36}{5} \right] = [3, 7.2]. \end{aligned}$$