Spring 2007 Math 152
Exam 3B: Problems and Solutions
Fri, 04/May ©2007, Art Belmonte

1. (d) Find the cosine of the angle θ between the vectors $\mathbf{v} = 2\mathbf{i} + 6\mathbf{j} - 4\mathbf{k}$ and $\mathbf{w} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

   - The cosine is

$$\begin{aligned}\cos\theta &= \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\,\|\mathbf{w}\|} \\ &= \frac{2+12-12}{\sqrt{4+36+16}\sqrt{1+4+9}} \\ &= \frac{2}{\sqrt{4\cdot 14^2}} = \frac{2}{2\,(14)} = \frac{1}{14}.\end{aligned}$$

2. (a) Find the vector projection of $\mathbf{b} = -\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$ onto $\mathbf{a} = -4\mathbf{i} + 8\mathbf{j} + 7\mathbf{k}$.

   - The vector projection is

$$\begin{aligned}\mathrm{proj}_{\mathbf{a}}\mathbf{b} &= \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}\right)\frac{\mathbf{a}}{\|\mathbf{a}\|} \\ &= \left(\frac{4-16-28}{\sqrt{16+64+49}}\right)\frac{[-4,8,7]}{\sqrt{129}} \\ &= -\frac{40}{129}\,[-4,8,7] \\ &= \frac{160}{129}\,\mathbf{i} - \frac{320}{129}\,\mathbf{j} - \frac{280}{129}\,\mathbf{k}.\end{aligned}$$

3. (c) The series $\displaystyle\sum_{n=2}^{\infty}\frac{\ln n}{n^2}$ converges by the Integral Test, as we now show.

   - First compute an antiderivative using integration by parts. Let $u = \ln x$ and $dv = x^{-2}\,dx$. Then $du = x^{-1}\,dx$ and $v = -x^{-1}$. Hence

$$\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} + \int x^{-2}\,dx = -\frac{(1+\ln x)}{x}.$$

   - Accordingly, $\displaystyle\int_2^{\infty}\frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\left(-\frac{1+\ln t}{t} + \frac{1+\ln 2}{2}\right) = \frac{1+\ln 2}{2}$ since $\displaystyle\lim_{t\to\infty}\frac{1+\ln t}{t} = \lim_{t\to\infty}\frac{1/t}{1} = 0$ by employing L'Hospital's Rule.

4. (e) Find the sum of the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-3)^{n-1}}{5^n}$.

   - The sum of geometric series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{5}\right)\left(-\frac{3}{5}\right)^{n-1}$ is $\displaystyle\frac{1/5}{1-\left(-\frac{3}{5}\right)} = \frac{1/5}{8/5} = \frac{1}{8}$.

5. (b) Choose the option that best describes the sequence $a_n = (-1)^n\,\dfrac{n}{n+2}$, $n \geq 1$.

   - The sequence is *bounded* since $|a_n| = \dfrac{n}{n+2} < \dfrac{n}{n} = 1$.
   - It is *nonmontonic* since its terms alternate in sign.

6. (a) Which of the following statements is true for the series $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln n}$?

   - *The series is convergent, but not absolutely convergent,* as we now show. (We say that the series is *conditionally convergent*.)
   - This alternating series converges by the Alternating Series Test since $b_n = |a_n| = \dfrac{1}{n\ln n} \downarrow 0$.
   - However, the corresponding series of absolute values $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ diverges by the Integral Test since $\displaystyle\int_2^{\infty}\frac{1}{x\ln x}\,dx = \lim_{t\to\infty}(\ln(\ln t) - \ln(\ln 2)) = \infty$.

7. (d) For what values of x does the power series $\displaystyle\sum_{n=0}^{\infty}\frac{n^2}{2^n}(x-3)^n$ converge?

   - The Root Test guarantees absolute convergence provided $\sqrt[n]{|a_n|} = \dfrac{\sqrt[n]{n^2}\,|x-3|}{2} \to \dfrac{|x-3|}{2} < 1$, whence $|x-3| < 2$. Thus $-2 < x-3 < 2$ or $1 < x < 5$. Alternatively, the Ratio Test also yields $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^2|x-3|^{n+1}}{2^{n+1}}\cdot\frac{2^n}{n^2|x-3|^n} = \frac{\left(1+\frac{1}{n}\right)^2|x-3|}{2} \to \frac{|x-3|}{2} < 1$.
   - For $x = 1$, the series is $\sum(-1)^n\,n^2$, which diverges by the Test for Divergence.
   - For $x = 5$, the series is $\sum n^2$, which also diverges by the Test for Divergence.
   - We conclude that the interval of convergence is $1 < x < 5$.

8. (d) Find $T_2(x)$, the second-degree Taylor polynomial for $f(x) = \sqrt{x+3}$ about $a = 1$.

   - Compute the first two derivatives of f.

$$\begin{aligned}f(x) &= (x+3)^{1/2} \\ f'(x) &= \tfrac{1}{2}(x+3)^{-1/2} \\ f''(x) &= -\tfrac{1}{4}(x+3)^{-3/2}\end{aligned}$$

   - Evaluate f and these derivatives at 1.

$$\begin{aligned}f(1) &= 2 \\ f'(1) &= \tfrac{1}{4} \\ f''(1) &= -\tfrac{1}{32}\end{aligned}$$

   - Write down the requested Taylor polynomial.

$$T_2(x) = \sum_{n=0}^{2}\frac{f^{(n)}(1)}{n!}(x-1)^n = 2 + \tfrac{1}{4}(x-1) - \tfrac{1}{64}(x-1)^2$$

9. The function $f(x) = \dfrac{x}{1+x^3}$ has a Taylor series expansion about $a = 1$ given by

$$f(x) = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + \cdots.$$

(a) Find these values: $f(1)$, $f'(1)$, and $f''(1)$.

- Compute the first two derivatives of f.

$$\begin{aligned} f(x) &= \frac{x}{1+x^3} \\ f'(x) &= \frac{\left(1+x^3\right)(1) - x\left(3x^2\right)}{\left(1+x^3\right)^2} = \frac{1-2x^3}{\left(1+x^3\right)^2} \\ f''(x) &= \frac{\left(1+x^3\right)^2\left(-6x^2\right) - \left(1-2x^3\right)\cdot 2\left(1+x^3\right)\left(3x^2\right)}{\left(1+x^3\right)^4} \end{aligned}$$

- Evaluate f and these derivatives at 1.

$$\begin{aligned} f(1) &= \frac{1}{2} \\ f'(1) &= -\frac{1}{4} \\ f''(1) &= \frac{-24+12}{16} = -\frac{12}{16} = -\frac{3}{4}. \end{aligned}$$

(b) Now compute these coefficients: a_0, a_1, and a_2.

- We have $a_n = \dfrac{f^{(n)}(1)}{n!}$ for $n \geq 0$.

$$\begin{aligned} a_0 &= \frac{1}{2} \\ a_1 &= -\frac{1}{4} \\ a_2 &= -\frac{3}{8} \end{aligned}$$

10. (a) Write down the Maclaurin series expansion for $\sin t$.

- We have $\sin t = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} t^{2n+1}$ for $t \in \mathbb{R}$.

(b) Use part (a) to find the power series expansion for the sine integral $\operatorname{Si}(x) = \displaystyle\int_0^x \frac{\sin t}{t}\,dt$.

- Therefore,

$$\begin{aligned} \operatorname{Si}(x) &= \int_0^x \frac{\sin t}{t}\,dt \\ &= \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} t^{2n}\,dt \\ & \left(\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!\,(2n+1)}\right)\Bigg|_0^x \\ &= \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!\,(2n+1)}\right) - \left(\sum_{n=0}^{\infty} 0\right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!\,(2n+1)} \text{ for } x \in \mathbb{R}. \end{aligned}$$

11. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-5)^{n+1}}{(2n+1)!}$.

(a) Prove that the series converges absolutely.

- The Ratio Test gives

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{5^{n+2}}{(2n+3)!} \cdot \frac{(2n+1)!}{5^{n+1}} = \frac{5}{(2n+2)(2n+3)} \to 0 < 1.$$

Hence the series converges absolutely.

(b) Use s_2, the second partial sum, to approximate the sum of the series.

- We have $s_2 = \dfrac{25}{6} - \dfrac{125}{120} = \dfrac{500-125}{120} = \dfrac{375}{120} = \dfrac{75}{24} = \dfrac{25}{8} = 3\frac{1}{8}$.

(c) Find an upper bound on the remainder in using s_2 to approximate the sum of the series.

- The Alternating Series Estimation Theorem yields

$$|R_2| \leq |a_3| = \frac{5^4}{7!} = \frac{625}{5040} = \frac{125}{1008} \approx 0.124,$$

any one of which is fine.

12. Find the power series expansion about $a = 0$ for $f(x) = \dfrac{1}{1+9x^2}$ and determine its radius of convergence.

- Use the Geometric Series Theorem!

$$\frac{1}{1-\left(-9x^2\right)} = \sum_{n=0}^{\infty} \left(-9x^2\right)^n = \sum_{n=0}^{\infty} (-1)^n 9^n x^{2n},$$

provided $\left|-9x^2\right| < 1$ or $|x| < \frac{1}{3}$.

- The radius of convergence is $R = \frac{1}{3}$.