

Spring 2008 Math 152
Exam 1B: Problems and Solutions
Mon, 18/Feb ©2008 Art Belmonte

1. (e) Evaluate $\int_0^{\pi/2} \sin^5 x \cos^3 x \, dx$.

- This is a trigonometric integral. Use the trig identity $\sin^2 x + \cos^2 x = 1$.

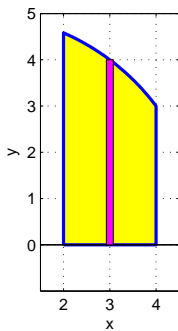
$$\begin{aligned} & \int_0^{\pi/2} \sin^5 x \cos^2 x \cos x \, dx \\ &= \int_0^{\pi/2} \sin^5 x (1 - \sin^2 x) \cos x \, dx \\ &= \int_0^{\pi/2} (\sin^5 x - \sin^7 x) \cos x \, dx \\ &= \left(\frac{1}{6} \sin^6 x - \frac{1}{8} \sin^8 x \right) \Big|_0^{\pi/2} \\ &= \left(\frac{1}{6} - \frac{1}{8} \right) - (0) \\ &= \frac{4-3}{24} = \frac{1}{24} \end{aligned}$$

[In the fourth step you may use the Substitution Rule.]

2. (a) Find the volume (in cm^3) of the solid obtained by rotating the region bounded by the given curves about the x -axis.

$$y = \sqrt{25 - x^2} \quad y = 0 \quad x = 2 \quad x = 4$$

- Here is a plot of the region.



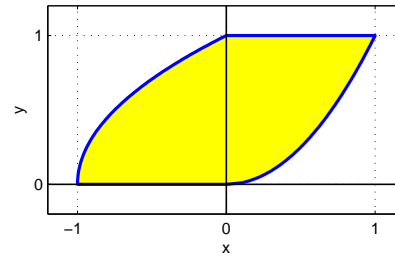
- The volume is

$$\begin{aligned} V &= \int \pi r^2 \, dx = \int \pi y^2 \, dx \\ &= \pi \int_2^4 (25 - x^2) \, dx \\ &= \pi \left(25x - \frac{1}{3}x^3 \right) \Big|_2^4 \\ &= \pi \left(100 - \frac{64}{3} \right) - \pi \left(50 - \frac{8}{3} \right) \\ &= \left(50 - \frac{56}{3} \right) \pi = \frac{94}{3} \pi \, \text{cm}^3. \end{aligned}$$

3. (c) Find the area (in cm^2) of the region in the xy -plane bounded by the following curves.

$$x = y^2 - 1 \quad x = \sqrt{y} \quad y = 0 \quad y = 1$$

- Here is a plot of the region.



- The area is

$$\begin{aligned} A &= \int_0^1 y^{1/2} - (y^2 - 1) \, dy \\ &= \left(\frac{2}{3}y^{3/2} - \frac{1}{3}y^3 + y \right) \Big|_0^1 \\ &= \left(\frac{2}{3} - \frac{1}{3} + 1 \right) - 0 = \frac{4}{3} \, \text{cm}^2. \end{aligned}$$

4. (b) A force of 30 N is required to maintain a spring stretched from its natural length of 12 cm to 15 cm. How much work (in joules) is done in stretching the spring from 12 cm to 20 cm?

- Via Hooke's Law, $30 = F = kx = k \left(\frac{15-12}{100} \right)$ implies $k = 1000$.
- The work is

$$\begin{aligned} W &= \int_0^{(20-12)/100} 1000x \, dx \\ &= 500x^2 \Big|_0^{8/100} \\ &= \frac{320}{100} - 0 = \frac{16}{5} \, \text{J}. \end{aligned}$$

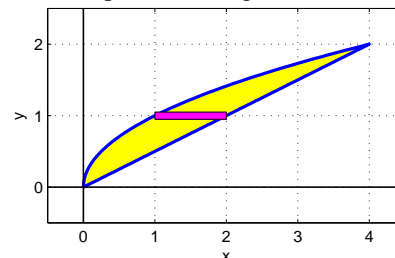
5. (d) Evaluate $\int x^3 \ln x \, dx$.

- Use integration by parts. Let $u = \ln x$ and $dv = x^3 \, dx$. Then $du = \frac{1}{x} \, dx$ and $v = \frac{1}{4}x^4$.
- Thus $\int x^3 \ln x \, dx = \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^3 \, dx = \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C$.

6. (c) Find the volume (in cm^3) of the solid obtained by rotating the region in the first quadrant bounded by the given curves about the y -axis.

$$x = y^2 \quad x = 2y$$

- When the curves intersect, their x -coordinates are equal. Thus $y^2 = 2y$ or $y(y - 2) = 0$ implies $y = 0, 2$. Here is a plot of the region.



- Via “washers,” the volume is

$$\begin{aligned}
 V &= \int \pi r_o^2 - \pi r_i^2 dy \\
 &= \pi \int_0^2 (2y)^2 - (y^2)^2 dy \\
 &= \pi \int_0^2 4y^2 - y^4 dy \\
 &= \pi \left(\frac{4}{3}y^3 - \frac{1}{5}y^5 \right) \Big|_0^2 \\
 &= \pi \left(\frac{32}{3} - \frac{32}{5} \right) - 0 \\
 &= 32\pi \left(\frac{5-3}{15} \right) = \frac{64}{15}\pi \text{ cm}^3.
 \end{aligned}$$

7. (d) Find the average value of $f(x) = \frac{3}{(1+x)^2}$ on $[1, 3]$.

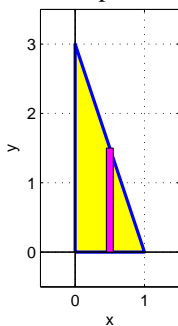
- The average value is

$$\begin{aligned}
 f_{ave} &= \frac{1}{3-1} \int_1^3 3(1+x)^{-2} dx \\
 &= \left(-\frac{3/2}{1+x} \right) \Big|_1^3 \\
 &= \left(-\frac{3}{8} \right) - \left(-\frac{3}{4} \right) = \frac{3}{8}.
 \end{aligned}$$

[In the second step you may use the Substitution Rule.]

8. (b) Find the volume (in cm^3) of the solid S whose base in the xy -plane is the triangular region with vertices $(0, 0)$, $(1, 0)$, and $(0, 3)$. Cross-sections perpendicular to the x -axis are squares.

- Here is a plot of the base.

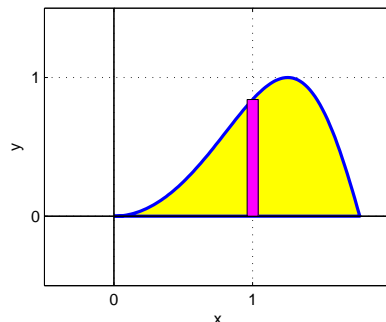


- The volume is

$$\begin{aligned}
 V &= \int y^2 dx \\
 &= \int_0^1 (3-3x)^2 dx \\
 &= \int_0^1 9 - 18x + 9x^2 dx \\
 &= (9x - 9x^2 + 3x^3) \Big|_0^1 \\
 &= 3 - 0 = 3 \text{ cm}^3.
 \end{aligned}$$

9. (e) Use the method of cylindrical shells to find the volume (in cm^3) generated by rotating the region in the first quadrant bounded by the x -axis and the curve $y = \sin(x^2)$, $0 \leq x \leq \sqrt{\pi}$, about the y -axis.

- Here is a plot of the region.



- The volume is

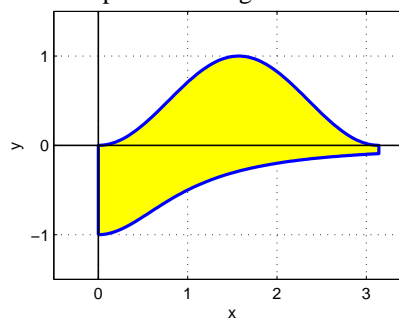
$$\begin{aligned}
 V &= \int 2\pi rh dx \\
 &= \pi \int_0^{\sqrt{\pi}} 2x \sin(x^2) dx \\
 &= \left(-\pi \cos(x^2) \right) \Big|_0^{\sqrt{\pi}} \\
 &= (\pi) - (-\pi) = 2\pi \text{ cm}^3.
 \end{aligned}$$

[In the third step you may use the Substitution Rule.]

10. Find the area (in cm^2) of the region in the xy -plane bounded by the following curves.

$$y = \sin^2 x \quad y = -\frac{1}{1+x^2} \quad x = 0 \quad x = \pi$$

- Here is a plot of the region.



- The area is

$$\begin{aligned}
 A &= \int_0^{\pi} \sin^2 x - \left(-\frac{1}{1+x^2} \right) dx \\
 &= \int_0^{\pi} \frac{1 - \cos 2x}{2} + \frac{1}{1+x^2} dx \\
 &= \left(\frac{x - \frac{1}{2} \sin 2x}{2} + \tan^{-1} x \right) \Big|_0^{\pi} \\
 &= \left(\frac{\pi}{2} + \tan^{-1} \pi \right) - 0 = \frac{\pi}{2} + \tan^{-1} \pi.
 \end{aligned}$$

[In the second step we used a trig identity.]

11. Evaluate $\int_0^1 x^5 e^{x^3} dx$.

- First use integration by parts to find an antiderivative $F(x)$ of the integrand. Let $u = x^3$ and $dv = x^2 e^{x^3} dx$. Then $du = 3x^2 dx$ and $v = \frac{1}{3} e^{x^3}$. Therefore,

$$\begin{aligned} \int x^5 e^{x^3} dx &= \frac{1}{3} x^3 e^{x^3} - \int x^2 e^{x^3} dx \\ F(x) &= \frac{1}{3} e^{x^3} (x^3 - 1). \end{aligned}$$

- Hence $\int_0^1 x^5 e^{x^3} dx = F(x)|_0^1 = 0 - \left(-\frac{1}{3}\right) = \frac{1}{3}$.

12. Evaluate $\int \frac{x^3 - 2x^2 - 4}{x^3 - 2x^2} dx$.

- The degree of the numerator is not less than that of the denominator. So first divide the numerator by the denominator to obtain $\int 1 + \frac{-4}{x^3 - 2x^2} dx = *$.
- Use the method of partial fractions on the second term.

$$\begin{aligned} \frac{-4}{x^2(x-2)} &= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} \\ -4 &= Ax(x-2) + B(x-2) + Cx^2 \\ -4 &= (A+C)x^2 + (-2A+B)x - 2B \end{aligned}$$

Equate like coefficients: $-2B = -4$ implies $B = 2$. Next, $-2A + B = 0$ implies $A = \frac{1}{2}B = 1$. Finally, $A + C = 0$ implies $C = -A = -1$.

- Accordingly,

$$\begin{aligned} * &= \int 1 + \frac{1}{x} + 2x^{-2} - \frac{1}{x-2} dx \\ &= x + \ln|x| - \frac{2}{x} - \ln|x-2| + C. \end{aligned}$$

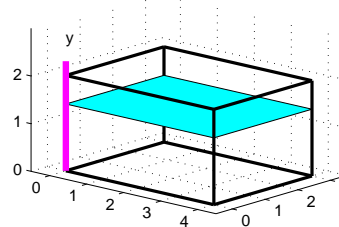
13. Evaluate $\int_{-3}^3 \frac{27}{(36-x^2)^{3/2}} dx$.

- First off, the integrand is even and the interval of integration is symmetric about 0. Thus the original integral is equivalent to $2 \int_0^3 \frac{27}{(36-x^2)^{3/2}} dx$. (This makes hand computation easier, although it is by no means necessary.)
- Use trigonometric substitution. Let $x = 6 \sin \theta$. Then $dx = 6 \cos \theta d\theta$.
- When $x = 0$, we have $\theta = 0$; whereas when $x = 3$, we have $\theta = \pi/6$. Therefore,

$$\begin{aligned} 2 \int_0^{\pi/6} \frac{27 \cdot 6 \cos \theta d\theta}{6^3 \cos^3 \theta} &= \frac{3}{2} \int_0^{\pi/6} \sec^2 \theta d\theta \\ &= \frac{3}{2} \tan \theta \Big|_0^{\pi/6} \\ [\text{Recall } \tan \theta &= \frac{\sin \theta}{\cos \theta}.] &= \left(\frac{3}{2}\right) \left(\frac{1/2}{\sqrt{3}/2}\right) - 0 \\ &= \frac{\sqrt{3}}{2}. \end{aligned}$$

14. An aquarium 4 m long, 3 m wide, and 2 m deep is filled with water. Find the amount of work (in joules) required to pump all the water out of the aquarium. The mass density of water is $\rho = 1000 \text{ kg/m}^3$ and the acceleration due to gravity is $g = 9.8 = \frac{98}{10} \text{ m/s}^2$. The answer is an integer. Yes, Citizen, simplify it *all the way!*

- Here is a diagram of the aquarium together with a typical differential layer of water within.



- Herewith the so-called “march of the differentials.” It shows the differential volume, mass, force (weight), as well as work required to lift the layer to the top of the aquarium.

$$dV = (4)(3) dy = 12 dy$$

$$dm = \rho dV = 12\rho dy$$

$$dF = (dm)g = 12\rho g dy$$

$$dW = (dF)(D) = (dF)(2-y) = 12\rho g(2-y) dy$$

- Integrate to find the total work done.

$$\begin{aligned} W &= \int dW = \int_0^2 12(1000)(98/10)(2-y) dy \\ &= 1200(100-2) \int_0^2 (2-y) dy \\ &= (120000 - 2400) \left(2y - \frac{1}{2}y^2\right) \Big|_0^2 \\ &= (120000 - 2400)(4 - 2) - 0 \\ &= 240000 - 4800 = 235,200 \text{ J} \end{aligned}$$