Spring 2008 Math 152
Exam 2B: Solutions
Mon, 24/Mar ©2008 Art Belmonte

1. (d) Determine the area of the surface (in cm^2) obtained by rotating the curve $x = y^2$, $0 \le y \le \sqrt{2}$, about the x-axis.

- The surface area is

$$\begin{aligned} S &= \int 2\pi r\, ds \\ &= \int_a^b 2\pi y \sqrt{1 + (dx/dy)^2}\, dy \\ &= 2\pi \int_0^{\sqrt{2}} y\sqrt{1+(2y)^2}\, dy \\ &= 2\pi \int_0^{\sqrt{2}} \left(1+4y^2\right)^{1/2} y\, dy \\ &= \left(\tfrac{2\pi}{8}\right)\left(\tfrac{2}{3}\right)\left(1+4y^2\right)^{3/2}\Big|_0^{\sqrt{2}} \\ &= \frac{\pi}{6}(27) - \frac{\pi}{6}(1) = \frac{26}{6}\pi = \frac{13}{3}\pi \approx 13.61 \text{ cm}^2. \end{aligned}$$

2. (e) Solve the initial value problem $\sec x \dfrac{dy}{dx} = e^{y+\sin x}$, $y(0) = 0$. Now, what is the value of $y(\pi/6)$?

- The equation $\sec x \frac{dy}{dx} = e^{y+\sin x}$ is *separable*.

$$\begin{aligned} e^{-y}\, dy &= e^{\sin x}\cos x\, dx \\ -e^{-y} &= e^{\sin x} + C \\ -1 &= 1 + C \quad [\text{Recall } y(0) = 0.] \\ C &= -2 \\ -e^{-y} &= e^{\sin x} - 2 \\ e^{-y} &= 2 - e^{\sin x} \\ -y &= \ln\left(2 - e^{\sin x}\right) \\ y &= -\ln\left(2 - e^{\sin x}\right) \\ y(\pi/6) &= -\ln\left(2 - e^{1/2}\right) = -\ln\left(2 - \sqrt{e}\right) \approx 1.05 \end{aligned}$$

3. (d) The speedometer reading v on a car was observed at 2-minute intervals and recorded in the chart below. Use the Trapezoidal Rule with $n = 5$ to estimate the distance traveled by the car during these 10 minutes; i.e., $\frac{1}{6}$ hour.

t (min)	0	2	4	6	8	10
v (mi/h)	40	45	52	56	57	56

- Now $\Delta t = h = \dfrac{b-a}{n} = \dfrac{\frac{1}{6} - 0}{5} = \dfrac{1}{30}$. Thus

$$T_5 = \frac{1}{30}\left(\frac{40+56}{2} + 45 + 52 + 56 + 57\right) = 8.6 \text{ miles.}$$

4. (c) Given masses 3, 4, and 5, located at points $(1, -4)$, $(0, 6)$, and $(-5, 3)$, respectively, find the center of mass of the system.

- Assign the data, then compute the center of mass.
- Let $\mathbf{p} = \begin{bmatrix} 3 & 4 & 5 \end{bmatrix}$ be a row vector of the masses and $\mathbf{r} = \begin{bmatrix} 1 & -4 \\ 0 & 6 \\ -5 & 3 \end{bmatrix}$ a matrix whose rows are position vectors of the coordinates. Let m be the sum of the masses: $3 + 4 + 5 = 12$.
- Then the position vector of the center of mass is given by CM $= [\bar{x}, \bar{y}] = \dfrac{1}{m}\mathbf{pr}$, as follows.

$$\begin{aligned} \frac{1}{m}\mathbf{pr} &= \frac{1}{12}\begin{bmatrix} 3 & 4 & 5 \end{bmatrix}\begin{bmatrix} 1 & -4 \\ 0 & 6 \\ -5 & 3 \end{bmatrix} \\ &= \frac{1}{12}[3 + 0 - 25, -12 + 24 + 15] \\ &= \frac{1}{12}[-22, 27] = \left[-\frac{11}{6}, \frac{9}{4}\right] \approx [-1.83, 2.25] \end{aligned}$$

5. (a) Find the arc length (in cm) of the curve $x = t^3$, $y = \frac{3}{2}t^2$, $0 \le t \le \sqrt{3}$.

- The arc length is

$$\begin{aligned} L &= \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt \\ &= \int_0^{\sqrt{3}} \sqrt{\left(3t^2\right)^2 + (3t)^2}\, dt \\ &= 3\int_0^{\sqrt{3}} \left(t^2+1\right)^{1/2} t\, dt \\ &= \left(\tfrac{3}{2}\right)\left(\tfrac{2}{3}\right)\left(t^2+1\right)^{3/2}\Big|_0^{\sqrt{3}} \\ &= 8 - 1 = 7 \text{ cm.} \end{aligned}$$

6. (b) Using the error bound for Simpson's Rule, $|E_S| \le \dfrac{M(b-a)^5}{180n^4}$, where $M = \max\limits_{a \le x \le b}\left|f^{(4)}(x)\right|$, what is the smallest value of n that guarantees the approximation for $\int_1^2 \ln x\, dx$ is accurate to within $\epsilon = 30^3 \times 10^{-8}$?

- Let $f(x) = \ln x$. Then

$$M = \max_{a \le x \le b}\left|f^{(4)}(x)\right| = \max_{1 \le x \le 2}\left|-6x^{-4}\right| = 6.$$

Hence

$$\begin{aligned}|E_S| \le \frac{M(b-a)^5}{180n^4} &\le \epsilon \\ \frac{6(1)^5}{180n^4} &\le 30^3 \times 10^{-8} \\ n^4 &\ge \frac{10^8}{30^4} \\ n &\ge \frac{10^2}{30} = \frac{100}{30} = 3\tfrac{1}{3}.\end{aligned}$$

So choose $n = 4$, the next even integer larger than $3\frac{1}{3}$.

7. (b) A pool 5 meters long, 2 meters wide, and 1 meter deep is filled with water to a depth of $\frac{1}{2}$ meter. If the weight density of water is $\delta = \rho g = 9800\ \text{N/m}^3$, find the hydrostatic force of the water on the bottom of the pool.

   - The pressure is $P = \delta\,(\text{depth}) = \frac{1}{2}\delta$. The hydrostatic force is $F = PA = \frac{1}{2}\delta\,(5)\,(2) = 5\delta = 5\,(9800) = 49{,}000$ N.

8. (c) Use the Comparison Theorem to determine whether the integral $\int_1^\infty \frac{1}{(x+1)\left(x^2+1\right)}\,dx$ is convergent or divergent.

   - For $x \ge 1$, we have $0 \le \frac{1}{(x+1)\left(x^2+1\right)} = \frac{1}{x^3+x^2+x+1} \le \frac{1}{x^3}$. Thus

$$\begin{aligned}0 \le \int_1^\infty \frac{1}{(x+1)\left(x^2+1\right)}\,dx &\le \int_1^\infty \frac{1}{x^3}\,dx \\ &= \lim_{b\to\infty}\int_1^b x^{-3}\,dx \\ &= \lim_{b\to\infty}\left(\frac{x^{-2}}{-2}\right)\Big|_1^b \\ &= \lim_{b\to\infty}\left(-\frac{1}{2b^2}+\frac{1}{2}\right) = \frac{1}{2}.\end{aligned}$$

   Hence $\int_1^\infty \frac{1}{(x+1)\left(x^2+1\right)}\,dx$ converges by the Comparison Theorem by comparison with $\int_1^\infty \frac{1}{x^3}\,dx$.

9. (d) Which of the following is an integrating factor for the differential equation $x\,\frac{dy}{dx} - 2y = \cos x$?

   - First put the differential equation in standard linear form, $\frac{dy}{dx} - \frac{2}{x}\,y = \frac{\cos x}{x}$. We then see that an integrating factor is $e^{\int -2/x\,dx} = e^{-2\ln x} = e^{\ln\left(x^{-2}\right)} = 1/x^2$.

10. An aquarium in a restaurant has an odd-shaped vertical window at its end that opens to the public view. The water level in the aquarium is level with the top of the window, which ranges from the bottom of the aquarium at its middle to 1 foot high at the top. The window is bounded by the curve $y = x^6$, $-1 \le x \le 1$, and the line $y = 1$. Find the hydrostatic force pushing against this vertical window. The weight density of water is $\delta = 62.5 = \frac{125}{2}\ \text{lb/ft}^3$.

   - The window is a vertical plate at variable depth. Build up the integral for hydrostatic force step by step.

$$\begin{aligned}P &= \delta\,(\text{depth}) = \delta\,(1-y) \\ dA &= 2x\,dy = 2y^{1/6}\,dy \\ dF &= P\,dA = 2\delta y^{1/6}\,(1-y)\,dy \\ F &= \int_0^1 2\left(\frac{125}{2}\right)\left(y^{1/6}-y^{7/6}\right)dy \\ &= 125\left(\tfrac{6}{7}y^{7/6} - \tfrac{6}{13}y^{13/6}\right)\Big|_0^1 \\ &= 125\left(\frac{78-42}{91}\right) - 0 \\ &= 125\left(\frac{36}{91}\right) = 125\left(\frac{4\times 9}{91}\right) = \frac{4500}{91} \approx 49.45\ \text{lb}\end{aligned}$$

11. First solve the initial value problem $2xy' + y = 6x$, $y(4) = 20$, $x > 0$. Then find the value of $y(16)$.

   - Put this *linear* equation into standard linear form (SLF).

$$y' + \frac{1}{2x}\,y = 3$$

   - Construct an integrating factor.

$$\mu = e^{\int \frac{1}{2x}\,dx} = e^{\frac{1}{2}\ln x} = e^{\ln\left(x^{1/2}\right)} = x^{1/2}$$

   - Multiply the SLF by μ.

$$\begin{aligned}x^{1/2}y' + \tfrac{1}{2}x^{-1/2}y &= 3x^{1/2} \\ \left(yx^{1/2}\right)' &= 3x^{1/2}\end{aligned}$$

   - Antidifferentiate, then isolate y.

$$\begin{aligned}yx^{1/2} &= 2x^{3/2} + C \\ y &= 2x + Cx^{-1/2}\end{aligned}$$

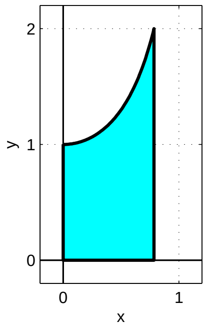
   - Resolve the initial condition $y(4) = 20$.

$$\begin{aligned}20 = y(4) &= 8 + \tfrac{1}{2}C \\ C &= \frac{20-8}{1/2} = 24 \\ y &= 2x + \frac{24}{\sqrt{x}}\end{aligned}$$

   - Finally, $y(16) = 32 + 6 = 38$.

12. Find the centroid (center of mass of a region of constant density k) of the region D in the xy-plane bounded by the curves

$$y = \sec^2 x \qquad y = 0 \qquad x = 0 \qquad x = \pi/4.$$



- The mass is

$$\begin{aligned} m &= \int_a^b k\left(f(x)-g(x)\right)\,dx \\ &= k\int_0^{\pi/4} \sec^2 x - 0\,dx \\ &= k\tan x|_0^{\pi/4} \\ &= k-0=k. \end{aligned}$$

- The moment about the y -axis is

$$\begin{aligned} M_y &= \int_a^b kx\left(f(x)-g(x)\right)\,dx \\ &= k\int_0^{\pi/4} x\left(\sec^2 x-0\right)\,dx \\ &= k\left(x\tan x+\ln|\cos x|\right)\Big|_0^{\pi/4} \\ &= k\left(\frac{\pi}{4}+\ln\frac{1}{\sqrt{2}}\right)-0=k\left(\frac{\pi}{4}+\ln\frac{1}{\sqrt{2}}\right). \end{aligned}$$

- The moment about the x-axis is

$$\begin{aligned} M_x &= \int_a^b k\cdot\tfrac{1}{2}\left(f^2(x)-g^2(x)\right)dx \\ &= \frac{k}{2}\int_0^{\pi/4} \sec^4 x-0\,dx \\ &= \frac{k}{2}\left(\frac{\tan^3 x}{3}+\tan x\right)\Bigg|_0^{\pi/4} \\ &= \frac{k}{2}\left(\frac{4}{3}\right)=\frac{2k}{3}. \end{aligned}$$

- Accordingly, the center of mass is

$$\begin{aligned} \text{CM}=[\bar{x},\bar{y}] &= \left[\frac{M_y}{m},\frac{M_x}{m}\right] \\ &= \left[\frac{k\left(\frac{\pi}{4}+\ln\frac{1}{\sqrt{2}}\right)}{k},\frac{\frac{2}{3}k}{k}\right] \\ &= \left[\frac{\pi}{4}+\ln\frac{1}{\sqrt{2}},\frac{2}{3}\right]\approx[0.44,0.67]. \end{aligned}$$

- ASIDE 1: To compute $\int x\sec^2 x\,dx=\star$, use integration by parts. Let $u=x$ and $dv=\sec^2 x$. Then $du=dx$ and $v=\tan x$. Accordingly, $\star=x\tan x-\int\frac{\sin x}{\cos x}\,dx=x\tan x+\ln|\cos x|$.
- ASIDE 2: To compute $\frac{1}{2}\int\sec^4\,dx$, use a trig identity.

$$\tfrac{1}{2}\int\left(\tan^2 x+1\right)\sec^2 x\,dx=\tfrac{1}{2}\left(\tfrac{1}{3}\tan^3 x+\tan x\right)$$

13. A tank initially contains 100 L of fresh water. A solution containing $\frac{2}{5}$ g/L of a chemical enters the tank at a rate of 5 L/min. The solution is kept mixed and is drained from the tank at the same rate.

    (a) Find an expression for $y(t)$, the amount of the chemical (in grams) in the tank after t minutes.

    (b) How much of the chemical is in the tank after 20 minutes?

- Let $y=y(t)$ be the amount of chemical in the tank at time t. Since the tank initially contains pure water, we have $y(0)=0$ g of the chemical in the tank at the start. The classical balance law gives

$$\begin{aligned} \frac{dy}{dt} &= \text{rate in}-\text{rate out} \\ y' &= \left(\tfrac{2}{5}\,\tfrac{\text{g}}{\text{L}}\times 5\,\tfrac{\text{L}}{\text{min}}\right) \\ &\quad -\left(\frac{y\text{ g}}{100\text{ L}}\times 5\,\frac{\text{L}}{\text{min}}\right) \\ y' &= 2-\tfrac{1}{20}y \\ y'+\tfrac{1}{20}y &= 2\ \text{[linear and separable]} \\ \mu &= e^{\int\frac{1}{20}\,dt}=e^{t/20} \\ e^{t/20}y'+\tfrac{1}{20}e^{t/20}y &= 2e^{t/20} \\ \left(ye^{t/20}\right)' &= 2e^{t/20} \\ ye^{t/20} &= 40e^{t/20}+C \\ y &= 40+Ce^{-t/20} \\ 0=y(0) &= 40+C \\ C &= -40 \\ y &= 40\left(1-e^{-t/20}\right) \end{aligned}$$

- Therefore, $y(20)=40\left(1-e^{-1}\right)\approx 25.28$ g.

14. Determine whether the integral $\int_1^e \frac{1}{x(\ln x)^2}\,dx$ is convergent or divergent. If it converges, determine its value. If it diverges, explain in detail why this is so.The integral diverges to ∞, as follows.

$$\begin{aligned} &\int_1^e \frac{1}{x(\ln x)^2}\,dx \\ &= \lim_{a\to 1^+}\int_a^e(\ln x)^{-2}\,\frac{1}{x}\,dx \\ &= \lim_{a\to 1^+}\left(-(\ln x)^{-1}\right)\Big|_a^e \\ &= \lim_{a\to 1^+}\left(-1+\frac{1}{\ln a}\right)=\infty \end{aligned}$$

Notes

Here is a Calc 3 way to compute the center of mass in Problem 12 via multiple integrals. It offers several advantages.

1. The formulation of $\bar{x}$ and $\bar{y}$ are symmetric.
2. It allows for variable density.
3. It naturally extends to a higher (or lower) number of dimensions.

- The mass of this region of constant density is

$$\begin{aligned} m = \iint_D \rho\, dA &= \int_0^{\pi/4} \int_0^{\sec^2 x} k\, dy\, dx \\ &= k \int_0^{\pi/4} y\Big|_0^{\sec^2 x} dx \\ &= k \int_0^{\pi/4} \sec^2 x\, dx \\ &= k \tan x\Big|_0^{\pi/4} \\ &= k - 0 = k. \end{aligned}$$

- The center of mass of the region is

$$\begin{aligned} [\bar{x}, \bar{y}] &= \frac{1}{m} \iint_D \rho\, [x, y]\; dA \\ &= \frac{1}{k} \int_0^{\pi/4} \int_0^{\sec^2 x} k\, [x, y]\; dy\, dx \\ &= \int_0^{\pi/4} \left[xy, \tfrac{1}{2} y^2\right]\Big|_{y=0}^{y=\sec^2 x} dx \\ &= \int_0^{\pi/4} \left[x \sec^2 x, \tfrac{1}{2} \sec^4 x\right] dx \\ &= \left[x \tan x + \ln|\cos x|\,, \tfrac{1}{2}\left(\tfrac{1}{3} \tan^3 x + \tan x\right)\right]\Big|_0^{\pi/4} \\ &= \left[\frac{\pi}{4} + \ln \frac{1}{\sqrt{2}},\ \frac{2}{3}\right] \approx [0.44, 0.67]\,. \end{aligned}$$