Spring 2003 Math 152 02/13/03

Common Exam 1B. Solutions.

PART 1. Multiple Choice (50 points)
Each problem is worth 5 points. Calculators are not allowed.

1. The velocity of a particle is given by $v(t) = te^{-t^2}$. What is the distance travelled during the first 3 seconds?

   A) $\frac{9}{2}e^{-9}$ B) $\frac{1}{2}(1-e^{-9})$ C) $2(1-e^{-9})$ D) $\frac{1}{2}e^{-9}$ E) $\frac{1}{2}(e^{-9}-1)$

   Solution. Since the velocity function is positive on [0,3], the distance travelled during the first 3 seconds is $s(3)-s(0) = \int_0^2 te^{-t^2}\,dt = -\frac{1}{2}e^{-t^2}\Big|_0^3 = -\frac{1}{2}(e^{-9}-1) = \frac{1}{2}(1-e^{-9})$. Answer: B

2. $\int_0^{\pi/4} \cos^2 x\,dx =$

   A) $\frac{\pi}{8}$ B) $\frac{\pi}{8}-\frac{1}{2}$ C) $\frac{\pi}{8}+\frac{1}{4}$ D) $\frac{\pi}{8}-\frac{1}{4}$ E) $\frac{\pi}{8}+\frac{1}{2}$

   Solution. $\int_0^{\pi/4} \cos^2 x\,dx = \int_0^{\pi/4} \frac{1+\cos 2x}{2}\,dx = \frac{1}{2}(x+\frac{1}{2}\sin 2x)\Big|_0^{\pi/4} = \frac{\pi}{8}+\frac{1}{4}$. Answer: C

3. $\int x^3 \ln x\,dx =$

   A) $3x^2\ln x - \ln x + C$ B) $\frac{x^4}{4}\ln x - \frac{x^4}{4} + C$ C) $x^4\ln x - x^4 + C$ D) $\frac{x^4}{4}\ln x - \frac{x^4}{16} + C$ E) $3x^2\ln x + x^2 + C$

   Solution. Integrate by parts with $u = \ln x$, $dv = x^3\,dx$. Then $du = \frac{1}{x}\,dx$, $v = \frac{1}{4}x^4$ and

   $\int x^3 \ln x\,dx = \frac{1}{4}x^4\ln x - \int \frac{1}{4}x^4(\frac{1}{x})\,dx = \frac{1}{4}x^4\ln x - \frac{1}{4}\int x^3\,dx = \frac{x^4}{4}\ln x - \frac{x^4}{16} + C$. Answer: D

4. $\int \frac{1}{(x+1)(x-2)}\,dx =$

   A) $\frac{1}{3}(\ln|x-2| - \ln|x+1|) + C$ B) $\ln|x+1| + \ln|x-2| + C$ C) $\frac{1}{3}(\ln|x+1| - \ln|x-2| + C$
   D) $\ln(x^2-x-2) + C$ E) $\tan^{-1}(x-1.5) + C$

   Solution. Decomposing $\frac{1}{(x+1)(x-2)}$ into partial fractions, we get $\frac{1}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$. It is easy to find that $A = -\frac{1}{3}$, $B = \frac{1}{3}$, so $\int \frac{1}{(x+1)(x-2)}\,dx = \frac{1}{3}\int -\frac{1}{x+1} + \frac{1}{x-2}\,dx = \frac{1}{3}(\ln|x-2| - \ln|x+1|) + C$. Answer: A

5. Using a trigonometric substitution $x = 2\sin\theta$, the integral $\int \frac{1}{x^2\sqrt{4-x^2}}\,dx$ becomes

A) $\frac{1}{2}\int \cot\theta \csc\theta \, d\theta$ B) $\frac{1}{8}\int \sec\theta \csc^2\theta \, d\theta$ C) $\frac{1}{4}\int \csc^2\theta \, d\theta$ D) $\frac{1}{4}\int \sec^2\theta \, d\theta$
E) $\frac{1}{2}\int \cos^2\theta \, d\theta$

Solution. $x = 2\sin\theta$, $x^2 = 2\sin^2\theta$, $\sqrt{4-x^2} = 2\cos\theta$, $dx = 2\cos\theta \, d\theta$. Then
$\displaystyle\int \frac{1}{x^2\sqrt{4-x^2}} dx = \int \frac{2\cos\theta \, d\theta}{4\sin^2\theta \cdot 2\cos\theta} = \frac{1}{4}\int \frac{1}{\sin^2\theta}, d\theta = \frac{1}{4}\int \csc^2\theta \, d\theta$. Answer. C

6. The average value of $y = \tan x$ on the interval from $x = 0$ to $x = \frac{\pi}{4}$ is

A) $\frac{1}{2}\ln 2$ B) $\frac{2}{\pi}$ C) $\frac{2}{\pi}\ln 2$ D) $\ln\frac{\sqrt{2}}{2}$ E) the average value does not exist

Solution. The average value of $y = \tan x$ on the interval from $x = 0$ to $x = \frac{\pi}{4}$ is

$$\frac{1}{\pi/4 - 0}\int_0^{\pi/4} \tan x \, dx = \frac{4}{\pi}\int_0^{\pi/4} \frac{\sin x}{\cos x} dx = \frac{4}{\pi}(-\ln\cos x)\Big|_0^{\pi/4} = \frac{4}{\pi}(-\ln\frac{\sqrt{2}}{2}) =$$
$$\frac{4}{\pi}(\ln\sqrt{2}) = \frac{4}{\pi}(\frac{1}{2}\ln 2 =)\frac{2}{\pi}\ln 2$$

Answer. C

7. The region bounded by $y = \sqrt{x}$, $y = 0$, $x = 2$ is revolved about the $x-$axis. The volume of the solid of revolution is

A) $\frac{4\sqrt{2}}{3}$ B) 6π C) $\frac{\pi}{2}$ D) 4π E) 2π

Solution. Using the method of disks, the volume is computed as $\pi\displaystyle\int_0^2 x \, dx = 2\pi$. Answer. E

8. The base of the solid is a circle $x^2 + y^2 = r^2$, and the cross-sections perpendicular to the $x-$axis are squares. The solid has volume

A) $\frac{8}{3}r^3$ B) $\frac{16}{3}r^3$ C) $4\pi r^3$ D) $2\pi r^3$ E) $\frac{8\pi}{3}r^3$

Solution. Take any x such that $-r \le x \le r$. Consider the base of the cross-section which is at the distance x from the center of the circle. Then the length of this base is $2\sqrt{r^2 - x^2}$. Hence the area of the cross-section (square) is $A(x) = 4(r^2 - x^2)$, and the volume of the solid is
$V = \int_{-r}^{r} A(x) \, dx = \int_{-r}^{r} 4(r^2 - x^2) \, dx = 8\int_0^r (r^2 - x^2) \, dx = 8(r^2 x - \frac{r^3}{3})\Big|_0^r = \frac{16}{3}r^3$. Answer. B

9. A force of 20 lb is required to hold a spring stretched from its natural length of 10 ft to 11 ft. The work done in stretching the spring from 10 ft to 12 ft is equal (in ft-lb) to

A) 4 B) 40 C) 120 D) 440 E) $\frac{40}{11}$

Solution. The relation between the force F and the deviation of the spring from its natural length, x, is $F = kx$. Since a force of 20 lb is required to hold a spring stretched by 1 ft from its natural length (from 10 ft to 11 ft), we conclude that $20 = k \cdot 1$, and so $k = 20$, $F(x) = 20x$. Next, since the work is $W = \int F(x)\,dx$, we get $W = \int_0^2 20x\,dx = 40$. Answer. B

10. If $\int_1^2 f(x)\,dx = 3$, $f(1) = 2$, $f(2) = -1$, $f'(1) = 4$, $f'(2) = -1$, then $\int_1^2 x\,f'(x)\,dx =$
A) -7 B) 6 C) -6 D) 1 E) -1

Solution. Integrate by parts with $u = x$, $dv = f'(x)\,dx$. Then $du = dx$, $v = \int f'(x) = f(x)$. Next, $\int_1^2 x\,f'(x)\,dx = x\,f(x)\Big|_0^2 - \int_0^2 f(x)\,dx = 2 \cdot f(2) - 1 \cdot f(1) - 3 = -7$. Answer. A

PART 2. Work-Out (50 points)

11. The region is bounded by the curves $y = x^2 - 3$ and $y = 2x$.

a) (2 points) Sketch the region.
Solution. The points of intersection of the curves are $(3, 6)$, $(-1, -2)$. (Solve $x^2 - 3 = 2x$ or $x^2 - 2x - 3 = 0$ to find the $x-$values of the points of intersection, then compute the $y-$values)

b) (8 points) Compute the area of the region.

Solution. The area of the region is equal to $\int_{-1}^3 (2x - x^2 + 3)\,dx = x^2 - \dfrac{x^3}{3} + 3x\Big|_{-1}^3 = \dfrac{32}{3}$

12. Consider the region bounded by $y = \sin x\; y = 0\; x = 0$ and $x = \dfrac{\pi}{2}$.
The region is revolved about the y-axis.
Find the volume of the solid of revolution.

a) (1 point) Sketch the region. The region is bounded by the graph of sine function from above, and the $x-$axis from below, $0 \le x \le \frac{\pi}{2}$

b) (9 points) Set up the integral representing the volume of the solid of revolution.
Indicate specifically the method used (cylindrical shells or disks/washers).
Compute the integral.

Solution. The method of shells works better here. The volume of the solid is equal to $2\pi \int_0^{\pi/2} x\,\sin x\,dx$.
To evaluate the integral, integrate by parts with $u = x$, $dv = sinx$. Then $du = dx$, $v = -\cos x$. Hence
$2\pi \int_0^{\pi/2} x\,\sin x\,dx = 2\pi(-x\,\cos x\Big|_0^{\pi/2} + \int_0^{\pi/2} \cos x\,dx) = 2\pi$

13. An aquarium 2 m long, $\dfrac{1}{2}$ m wide, and 1 m deep is full of water. Find the work needed to pump half of the water out of the aquarium. (Use the facts that the density of water is $\rho = 1000$ kg/m^3 and that the acceleration due to gravity is $g = 9.8$ m/s^2.)

Solution. Choose the $x-$axis going perpendicular to the base of the aquarium through one of the corners and the origin at the top corner, positive direction-downward. Then take a slice at the level x from the top, with thickness Δx. The volume of this slice is $2 \cdot \dfrac{1}{2} \cdot \Delta x = \Delta x$.

The weight of the slice is $\Delta x \cdot \rho \cdot g = \rho\, g \Delta x$. The distance the slice needs to travel to reach the top is x. Since only half of water is needed to be pumped out, $0 \le x \le \frac{1}{2}$. Hence the work is

$$\int_0^{\frac{1}{2}} \rho\, g\, x\, dx = \rho\, g \frac{x^2}{2}\Big|_0^{1/2} = \frac{\rho\, g}{8} = 1225\text{Joules}$$

An alternative solution. Choose the origin at the bottom corner and positive direction upward. A slice at the level x from the bottom has the same weight as before, but the distance to the top is $1 - x$ and the integral for work is

$$\int_{\frac{1}{2}}^1 \rho\, g\, (1-x)\, dx = \rho\, g\, (x - \frac{x^2}{2})\Big|_{1/2}^1 = \frac{\rho\, g}{8} = 1225\text{Joules}$$

14. Compute the integral. Show all necessary steps leading to an answer.

$\int \frac{\cos^3 x}{\sin^2 x}\, dx.$

Solution. First, $\cos^3 x = \cos^2 x \cdot \cos x = (1 - \sin^2 x) \cos x$. Substitute $u = \sin x$, $du = \cos x\, dx$. Then

$\int \frac{\cos^3 x}{\sin^2 x}\, dx = \int \frac{1 - \sin^2 x}{\sin^2 x} \cos x\, dx = \int (u^{-2} - 1)\, du = -u^{-1} - u + C = -\frac{1}{\sin x} - \sin x + C = -\csc x - \sin x + C.$

15. Compute the integral. Show all necessary steps leading to an answer.

$\int_2^4 \frac{\sqrt{x^2 - 4}}{x}\, dx.$

Solution. Substitute $x = 2 \sec\theta$. Then $dx = 2 \sec\theta \tan\theta\, d\theta$. If $x = 2 = 2\sec\theta$, then $\sec\theta = 1$, $\theta = 0$. If $x = 4 = 2\sec\theta$, then $\sec\theta = 2$, $\theta = \frac{\pi}{3}$.

Next, $\tan^2\theta = sec^2\theta - 1$, thus $x^2 - 4 = 4\sec^2\theta - 4 = 4\tan^2\theta$.

Hence $\int_2^4 \frac{\sqrt{x^2-4}}{x}\, dx = \int_0^{\pi/3} \frac{2\tan\theta}{2\sec\theta} 2\sec\theta \tan\theta\, d\theta = \int_0^{\pi/3} 2\tan^2\theta\, d\theta = 2\int_0^{\frac{\pi}{3}} (\sec^2\theta - 1)\, d\theta = 2(\tan\theta - \theta)\Big|_0^{\pi/3} = 2(\sqrt{3} - \frac{\pi}{3})$