# Common Exam 3 A. Solutions.

**PART 1. Multiple Choice (50 points)**

1. Which of the following sequences diverges?

   A) $\left\{\frac{3}{n}\right\}$ B) $\left\{\frac{(-1)^{n+1}}{3n+1}\right\}$ C) $\left\{\frac{2^n}{e^n}\right\}$ D) $\left\{\frac{n}{\ln n}\right\}$ E) $\left\{\frac{2n^2}{\sqrt{n^4+3}}\right\}$

   Solution. The first three sequences converge (to zero), the last one converges to 2. We use L'Hospital's Rule to show that

   $\lim\limits_{x\to\infty}\frac{x}{\ln x} = \lim\limits_{x\to\infty}\frac{1}{(\ln x)'} = \lim\limits_{x\to\infty}\frac{x}{1} = \infty$, so the sequence $\left\{\frac{n}{\ln n}\right\}$ diverges. Answer. D

2. Which of the following statements is true:

   a) If $a_n > 0$ and $\sum\limits_{n=1}^{\infty}(-1)^n a_n$ converges, then $\sum\limits_{n=1}^{\infty} a_n$ converges. False.

   b) If $a_n > 0$ and $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n} = \frac{4}{3}$, then $\sum\limits_{n=1}^{\infty} a_n$ converges. False (Ratio Test).

   c) The series $\sum\limits_{n=1}^{\infty}\frac{1}{n^{1.01}}$ converges. True. $p-$series, with $p = 1.01 > 1$

   d) The series $\sum\limits_{n=1}^{\infty}\frac{n+1}{\sqrt{n^4+10}}$ converges. False. Compare with harmonic series.

   e) If $\lim\limits_{n\to\infty} a_n = 0$, then $\sum\limits_{n=1}^{\infty}(-1)^n a_n$ converges.

   False. The condition $\lim\limits_{n\to\infty} a_n = 0$ is not sufficient for convergence.

   Answer. C.

3. Which of the following statements is true:

   a) If $\sum\limits_{n=1}^{\infty} a_n x^n$ converges at $x = -1.1$, then it also converges at $x = 7$.

   This statement is false; the series converges at every point $x$ such that $|x| \leq 1.1$

   b) If $\sum\limits_{n=1}^{\infty} a_n x^n$ diverges at $x = -2$, then it also diverges at x=3.

   This statement is true; the series diverges at every point $x$ such that $|x| \geq 2$

   c) $\sum\limits_{n=1}^{\infty}\frac{x^n}{n}$ converges at $x = 1$

   This statement is false; harmonic series diverges.

   d) $1 + \frac{(\pi/10)^2}{2!} + \frac{(\pi/10)^4}{4!} + ... + \frac{(\pi/10)^{2n}}{(2n)!} + .... = \cos\frac{\pi}{10}$

   This statement is false;

   $\cos\frac{\pi}{10} = 1 - \frac{(\pi/10)^2}{2!} + \frac{(\pi/10)^4}{4!} + ... + \frac{(-1)^{-n}(\pi/10)^{2n}}{(2n)!} + .... = \sum\limits_{n=1}^{\infty}\frac{(-1)^{-n}(\pi/10)^{2n}}{(2n)!}$

   e) $1 - \frac{(1/2)}{1!} + \frac{(1/2)^2}{2!} - ... + \frac{(-1)^n(1/2)^n}{n!} + .... = \sqrt{e}$

This statement is false; $\sqrt{e} = e^{1/2} = 1 + \frac{(1/2)}{1!} + \frac{(1/2)^2}{2!} + ... + \frac{(1/2)^n}{n!} + .... = \sum_{n=1}^{\infty} \frac{(1/2)^n}{n!}$

Answer. B

4. Choose the series $\sum_{n=1}^{\infty} a_n$ that diverges because $\lim_{n\to\infty} a_n \neq 0$:

A) $\sum_{n=1}^{\infty} \frac{(-1)^n}{5n^2+1}$ B) $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ C) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ D) $\sum_{n=1}^{\infty} \frac{8^{-n}}{3^n}$ E) $\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+1}\right)$

Solution. In the first four cases $\lim_{n\to\infty} a_n = 0$, while $\lim_{n\to\infty} \ln\left(\frac{n}{2n+1}\right) = \ln\left(\frac{1}{2}\right) \neq 0$.

Answer. E

5. Which of the following is true for the series $\sum_{n=1}^{\infty} (e^{\frac{1}{n}} - e^{\frac{1}{n+1}})$:

A) the series converges to 1 B) the series converges to $e$ C) the series converges to $e-1$
D) the series diverges to $\infty$ E) the series diverges to $-\infty$

Solution. The series $\sum_{n=1}^{\infty} (e^{\frac{1}{n}} - e^{\frac{1}{n+1}})$ is a telescoping series; the n-th partial sum is equal to $s_n = \sum_{k=1}^{n} (e^{\frac{1}{k}} - e^{\frac{1}{k+1}}) = e^1 - e^{\frac{1}{n+1}}$ so $\lim_{n\to\infty} s_n = e-1$, which means that the series converges to $e-1$. Answer. C

6. The sum of the series $3 - 1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - ...$ is

A) $\frac{7}{3}$ B) $\frac{9}{4}$ C) 2 D) $\frac{5}{2}$ E) $\frac{7}{4}$

Solution. Starting from the second term, the series is a geometric series. Computing the sum of the geometric series with the first term $a = -1$ and the ratio $r = -\frac{1}{2}$, we get $S = \frac{-1}{1+1/2} = -\frac{2}{3}$. Thus the sum of the given series is $3 - \frac{2}{3} = \frac{7}{3}$. Answer. A

7. Which of the following is true for the series $\sum_{n=1}^{\infty} \frac{\sin^3 n}{n\sqrt{n}}$:

A) the series converges absolutely by comparison with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

B) the series converges absolutely by comparison with $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

C) the series diverges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$

D) the series diverges by comparison with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

E) the series converges but not absolutely.

Solution. Since $\left|\frac{\sin^3 n}{n\sqrt{n}}\right| \leq \frac{1}{n\sqrt{n}}$, and since the series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges, the series $\sum_{n=1}^{\infty} \frac{|\sin^3 n|}{n^{3/2}}$ converges. Thus the given series converges absolutely. Answer. B

8. The coefficient of $x^2$ in the Maclaurin series for $e^{\sin x}$ is

A) $\frac{1}{4}$ B) 1 C) $-1$ D) 0 E) $\frac{1}{2}$

Solution. The Maclaurin series for the function $f(x) = e^{\sin x}$ can be obtained as
$e^{\sin x} = 1 + (\sin x) + \frac{(\sin x)^2}{2!} + ... + \frac{(\sin x)^n}{n!} + ... =$
$1 + (x - \frac{x^3}{3!} + ... + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + ...) + \frac{1}{2!}(x - \frac{x^3}{3!} + ... + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + ...)^2 + ...$
$+ \frac{1}{n!}(x - \frac{x^3}{3!} + ... + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + ...)^n + ....$

From this expansion we see that the coefficient of $x^2$ is $\frac{1}{2}$. Answer. E

9. The power series for $f(x) = \dfrac{1}{1+x^2}$ is

A) $1 + x^2 + x^4 + ... + x^{2n} + ...$

B) $x^2 + x^4 + ... + x^{2n} + ...$

C) $1 + 2x + 4x^2 + ... + (2^n)x^{2n} + ...$

D) $1 - x^2 + x^4 - ... + (-1)^n x^{2n} + ...$

E) $1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - ... + (-1)^n \dfrac{x^{2n}}{(2n)!} + ...$

Solution. Since $f(x) = \dfrac{1}{1+x^2}$ can be interpreted as the sum of the geometric series with the first term $a = 1$ and the ratio $r = -x^2$, the power series representation of the function $f$ is $f(x) = \dfrac{1}{1+x^2} = 1 - x^2 + x^4 - ... + (-1)^n x^{2n} + ....$ Answer. D

10. Compute $\lim\limits_{x \to 0} \dfrac{\cos(x^3) - 1}{x^6}$ (Hint: use Maclaurin series). The limit is equal to

A) $-\frac{1}{2}$ B) $\frac{1}{2}$ C) $\frac{1}{6}$ D) $-\frac{1}{6}$ E) $-\frac{1}{24}$

Solution. Using the Maclaurin series for $\cos(x^3) - 1$, we have $\lim\limits_{x \to 0} \dfrac{\cos(x^3) - 1}{x^6} = \lim\limits_{x \to 0} \dfrac{1}{x^6}(1 - \dfrac{(x^3)^2}{2!} + \dfrac{(x^3)^4}{4!} + ... - 1) = \lim\limits_{x \to 0} \dfrac{1}{x^6}(-\dfrac{(x^3)^2}{2!} + \dfrac{(x^3)^4}{4!} + ...) = -\dfrac{1}{2}$. Answer. A

**PART 2. Work-Out (50 points)**

11. a) (5 points) Determine whether the sequence $\left\{\dfrac{3+n}{n\sqrt{n}}\right\}$ converges or diverges. If it converges, find the limit. If it diverges, explain the reason.

Answer. Since $\lim\limits_{n \to \infty} \dfrac{3+n}{n\sqrt{n}} = 0$, the sequence $\left\{\dfrac{3+n}{n\sqrt{n}}\right\}$ converges to zero.

b) (5 points) Determine whether the series $\sum\limits_{n=1}^{\infty} \dfrac{3+n}{n\sqrt{n}}$ converges or diverges. Justify your answer.

Answer. Comparing the series $\sum\limits_{n=1}^{\infty} \dfrac{3+n}{n\sqrt{n}}$ with the divergent $p-$series $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{n}}$ $(p = \frac{1}{2} < 1)$, we conclude that the given series diverges. (Use the Limit Comparison Test to show that $\lim\limits_{n \to \infty} (\dfrac{3+n}{n\sqrt{n}}) : (\dfrac{1}{\sqrt{n}}) = 1$, which means that the series behave the same way).

12. (10 points) Determine whether the following series converges absolutely, converges but not absolutely, or diverges. NAME the test used, SHOW all work, and clearly STATE your conclusion.

a) (5 points)

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{(n+1)!}$$

Solution. Consider the corresponding series of the absolute values, $\sum_{n=1}^{\infty} \frac{3^n}{(n+1)!}$. Here $a_n = \frac{3^n}{(n+1)!}$, $a_{n+1} = \frac{3^{n+1}}{(n+2)!}$. Then $\lim_{x\to\infty} \frac{a_{n+1}}{a_n} = \lim_{x\to\infty} \frac{3^{n+1}(n+1)!}{3^n(n+2)!} = \lim_{x\to\infty} \frac{3}{n+2} = 0 < 1$. Thus, by the Ratio Test, the series $\sum_{n=1}^{\infty} \frac{3^n}{(n+1)!}$ converges, so the original series $\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{(n+1)!}$ converges absolutely.

b) (5 points) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n\,(\ln n)^{1/3}}$

Solution. Consider the corresponding series of the absolute values, $\sum_{n=2}^{\infty} \frac{1}{n\,(\ln n)^{1/3}}$. Applying the Integral Test, we have $\int_2^{\infty} \frac{1}{x\,(\ln x)^{1/3}} = \frac{2}{3}(\ln x)^{2/3}|_2^{\infty} = \infty$. This means that the series of the absolute values diverges; thus the original series does not converge absolutely. The question remains, though, whether it diverges or converges but not absolutely (converges conditionally). We apply the Alternating Series Test. The sequence $a_n = \frac{1}{n\,(\ln n)^{1/3}}$ is decreasing since $a_n = \frac{1}{n\,(\ln n)^{1/3}} > a_{n+1} = \frac{1}{(n+1)\,(\ln n+1)^{1/3}}$. Moreover, the sequence $a_n$ converges to 0, since $\lim_{x\emptyset\infty} \frac{1}{n\,(\ln n)^{1/3}} = 0$. Thus the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\,(\ln n)^{1/3}}$ converges.

Answer. The series converges but not absolutely.

13. (10 points) Find the radius and the interval of convergence of the power series. (Do not forget to check for convergence at the endpoints).

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x-3)^n}{\sqrt{n}\,4^n}$$

Solution. Consider the corresponding series of absolute values, $\sum_{n=1}^{\infty} \frac{|x-3|^n}{\sqrt{n}\,4^n}$. Here $a_n = \frac{|x-3|^n}{\sqrt{n}\,4^n}$, $a_{n+1} = \frac{|x-3|^{n+1}}{\sqrt{n+1}\,4^{n+1}}$. Then $\lim_{x\to\infty} \frac{a_{n+1}}{a_n} = \frac{|x-3|^{n+1}\sqrt{n}\,4^n}{|x-3|^n\sqrt{n+1}\,4^{n+1}} = \lim_{x\to\infty} \frac{|x-3|}{4}$. Thus, by the Ratio Test, the series $\sum_{n=1}^{\infty} \frac{|x-3|^n}{\sqrt{n}\,4^n}$ converges if $\frac{|x-3|}{4} < 1$ or if $|x-3| < 4$. This means that the radius of convergence is 4, so the original series $\sum_{n=1}^{\infty} \frac{(-1)^n (x-3)^n}{\sqrt{n}\,4^n}$ converges

absolutely if $|x-3|<4$ or if $-4<x-3<4$, $-1<x<7$.
Now we must test the convergence at the endpoints of this interval.
If $x=-1$, the series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(-4)^n}{\sqrt{n}\,4^n}=\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n\,4^n}{\sqrt{n}\,4^n}=\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$, which diverges as a $p$- series with $p=\frac{1}{2}<1$.
If $x=7$, the series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(4)^n}{\sqrt{n}\,4^n}=\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$, which converges by the Alternating series Test.
Therefore, the power series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(x-3)^n}{\sqrt{n}\,4^n}$ converges when $-1<x\leq 7$ (this is the interval of convergence), and diverges outside of this interval.

14. (10 points) Find the first four nonzero terms of the Taylor series expansion of the function $f(x)=\ln x$ at the point $x=2$.
Solution. We compute $f(x)=\ln x$, $f(2)=\ln 2$, $f'(x)=\frac{1}{x}$, $f'(2)=\frac{1}{2}$, $f''(x)=-\frac{1}{x^2}$, $f''(2)=-\frac{1}{4}$, $f'''(x)=\frac{2}{x^3}$, $f'''(2)=\frac{1}{4}$. Thus the required expansion of the function $\ln x$ at the point $x=2$ is $\ln 2+\frac{1}{2}(x-2)-\frac{1}{4\cdot 2!}(x-2)^2+\frac{1}{4\cdot 3!}(x-2)^3$.

15. a) ( 7 points) Use the first two nonzero terms in the Maclaurin series expansion of the function $f(x)=\dfrac{e^{-x^2}-1}{x}$ to approximate the value of the integral: $\displaystyle\int_0^{0.1}\frac{e^{-x^2}-1}{x}\,dx$.
Solution. First we find the Maclaurin series for $f(x)=\dfrac{e^{-x^2}-1}{x}$ by replacing $x$ with $-x^2$ in the series for $e^x$, then we subtract 1 and divide by $x$:
$f(x)=\frac{e^{-x^2}-1}{x}=\frac{1}{x}[\left(\sum_0^{\infty}\frac{(-x^2)^n}{n!}\right)-1]=\sum_1^{\infty}\frac{(-1)^n\,x^{2n-1}}{n!}=-x+\frac{x^3}{2!}-\frac{x^5}{3!}\cdots$
Next, integrating the power series term by term, we get $\displaystyle\int_0^{0.1}\frac{e^{-x^2}-1}{x}\,dx=\left(\sum_1^{\infty}\frac{(-1)^n\,x^{2n}}{(2n)n!}\right)\Big|_0^{0.1}=$
$\displaystyle(-\frac{x^2}{2}+\frac{x^4}{(4)2!}+\frac{x^6}{(6)3!}...)\Big|_0^{0.1}\approx-\frac{(0.1)^2}{2}+\frac{(0.1)^4}{(4)2!}\approx 5.01\times 10^{-3}=0.00501$
b) (3 points) Estimate the error of the approximation.
Solution. Since the series obtained by integration is an alternating series, we have $|R|<\frac{(0.1)^6}{(6)3!}\approx 2.78\times 10^{-8}$. Moreover, we can tell that the error is positive, so the estimate we obtained in a) underestimates the actual value of the integral, but by not more then $\frac{(0.1)^6}{(6)3!}\approx 2.78\times 10^{-8}$