Fall 2003
Math 152
COMMON EXAM 3
Test Form B

PRINT: Last Name: ______________________________ **First Name:** ______________________

Signature: ____ **ID:** __________________________

Instructor's Name: ____ **Section #** _____________

Instructor use only.

Multiple choice	
Q11	
Q12	
Q13	
Q14	
Q15	
Total	

INSTRUCTIONS

1. In **Part I** (Problems 1–10), mark the correct choice on your ScanTron form using a #2 pencil. *For your own records, also record your choices on your exam!* The ScanTrons will NOT be returned. Write your name and the color of your test on the ScanTron.

2. In **Part II** (Problems 11–15), write all solutions in the space provided. You may use the back of any page for scratch work, but all work to be graded must be shown in the space provided. CLEARLY INDICATE YOUR FINAL ANSWERS.

3. Turn off all electronic devices including cell phones.

PART I: MULTIPLE-CHOICE PROBLEMS

Each problem is worth 5 points: *NO* partial credit will be given. Calculators may *NOT* be used on this part.

1. Which of the statements about convergence of $\sum_{n=1}^{\infty} a_n, \quad a_n \geq 0,$ are true?

   (1) If $\lim_{n\to\infty} a_n = 0$ then the series converges,

   (2) If $a_n \geq \frac{1}{n^2}$ then the series converges,

   (3) If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ then the series diverges,

   (4) If $a_n \leq \frac{1}{n}$ then the series converges.

   (a) (4) only

   (b) All

   (c) None

   (d) (1) only

   (e) (2) and (3) only

   None are true.(c)

2. The series $\sum_{n=1}^{\infty} \frac{n^n}{(n!)^2}$

   (a) converges because $a_n \to 0$

   (b) diverges by the ratio test

   (c) converges by the ratio test

   (d) diverges by the comparison test

   (e) diverges by the integral test

   $\left(\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e \quad \text{could be useful}\right)$

   By the ratio test, $\lim \frac{a_{n+1}}{a_n} = \lim \frac{(n+1)^{n+1}}{(n+1)!(n+1)!} \frac{n!n!}{n^n} = \lim \frac{(n+1)^{n+1}}{n^n(n+1)(n+1)} = \lim \frac{(n+1)^n}{n^n(n+1)} = \lim \left(1+\frac{1}{n}\right)^n \frac{1}{n+1} = 0$ so (c) is correct.

3. The interval of convergence of $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt{n}}$ is

bf (a) $(-1, 1]$
(b) $[-1, 1]$
(c) $(-\infty, \infty)$
(d) $[-1, 1)$
(e) $(-1, 1)$

The ratio test gives convergence for $|x| < 1$. At $x = 1$ we have $\Sigma \frac{(-1)^n}{\sqrt{n}}$ which converges by the AST. At $x = -1$ we have $\Sigma \frac{1}{\sqrt{n}}$ which diverges, p-series, $p < 1$. So (a) is correct.

4. The series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n+1}{n}$ is

(a) convergent by the limit comparison test
(b) absolutely convergent
(c) convergent but not absolutely convergent
(d) divergent by the alternating series test
(e) divergent because $a_n \not\to 0$

$\lim \frac{n+1}{n} = \lim \left(1 + \frac{1}{n}\right) = 1$, so (e) is correct.

5. Consider the series (1) $\sum_{n=1}^{\infty} \frac{n^2+1}{2^n}$ and (2) $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{n^2+1}$

(a) At least one cannot be decided

(b) Both converge

(c) Both diverge

(d) (1) converges and (2) diverges

(e) (1) diverges and (2) converges

(1) converges by the ratio test and (2) diverges by comparsion with $\Sigma\frac{1}{\sqrt{n}}$ using the limit comparison test. So (d) is correct.

6. Find $\lim_{x\to 0} \frac{\cos(x^3)-1}{\sin(x^2)-x^2}$ (Maclaurin series are useful here)

(a) 3

(b) -1

(c) 0

(d) 1

(e) 2

$$\lim_{x\to 0} \frac{\left(1-\frac{x^6}{2}+\cdots\right)-1}{\left(x^2-\frac{x^6}{6}+\cdots\right)-x^2} = \lim_{x\to 0} \frac{-\frac{x^6}{2}+\cdots}{-\frac{x^6}{6}+\cdots}$$

$$= \frac{1}{2} \div \frac{1}{6} = 3. \quad \text{(a) is correct}$$

7. Find the limit of the sequence $\left\{ \dfrac{\ln(2n)}{\ln(n+1)} \right\}_{n=1}^{\infty}$.

(a) does not exist

(b) 2

(c) $\ln 2$

(d) 0

(e) 1

By L'Hopital, $\lim_{x\to\infty} \dfrac{\ln(2x)}{\ln(x+1)} = \lim_{x\to\infty} \dfrac{2}{2x} \Big/ \dfrac{1}{x+1} = \lim_{x\to\infty} \dfrac{x+1}{x} = \lim_{x\to\infty} 1 + \dfrac{1}{x} = 1$. (e) is correct.

8. The series $\sum_{n=1}^{\infty} \dfrac{(n!)^2}{(2n)!}$

(a) Diverges because $a_n \to \infty$

(b) Converges by the ratio test

(c) Diverges by the ratio test

(d) Diverges by the comparison test

(e) Diverges by the integral test

Apply the ratio test.

$\lim_{n\to\infty} \dfrac{(n+1)!(n+1)2n!}{(2n+2)!n!n!} = \lim_{n\to\infty} \dfrac{(n+1)(n+1)}{(2n+2)(2n+1)} = \lim_{n\to\infty} \dfrac{n^2+2n+1}{4n^2+6n+2} = \dfrac{1}{4}$. (b) is correct.

9. The series $\sum_{n=1}^{\infty} \frac{2^n}{\ln(n+1)}$

(a) Diverges by comparison to $\sum_{n=1}^{\infty} 2^n$

(b) Converges by the integral test

(c) Diverges by the ratio test

(d) Converges because $a_n \to 0$

(e) Converges by comparison to $\sum_{n=1}^{\infty} \frac{1}{n^2}$

The ratio test gives $\lim_{n\to\infty} \frac{2\ln(n+1)}{\ln(n+2)} = \lim_{n\to\infty} \frac{2}{n+1}\Big/\frac{1}{n+2} = 2$ so diverges. (c) is correct.

10. The sum of the series $\frac{2}{9} + \frac{2}{9^2} + \frac{2}{9^3} + \cdots$ is

(a) it diverges

(b) $\frac{1}{3}$

(c) $\frac{1}{4}$

(d) 1

(e) 2

First term is $\frac{2}{9}$, common ratio is $\frac{1}{9}$ so the sum is $\frac{2}{9}\Big/\left(1-\frac{1}{9}\right) = \frac{1}{4}$. (c) is correct

PART II. WORKOUT PROBLEMS

Each problem is worth 10pts. Calculators are NOT allowed on this part Partial credit will be given.

11. Find the first three terms in the Maclaurin series of $e^x \cos x$.

$$\begin{aligned} e^x \cos x &= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) \\ &= 1 + x + x^2\left(\frac{1}{2} - \frac{1}{2}\right) + x^3\left(\frac{1}{6} - \frac{1}{2}\right)\cdots \\ &= 1 + x - \frac{1}{3}x^3\cdots \end{aligned}$$

12. Find the full Taylor series of $f(x) = x^3$ for $a = 2$.

$$\begin{aligned} f(x) &= x^3, & f(2) &= 8 \\ f'(x) &= 3x^2, & f'(2) &= 12 \\ f''(x) &= 6x, & f''(2) &= 12 \\ f'''(x) &= 6, & f'''(2) &= 6 \\ f''''(x) &= 0, & f''''(2) &= 0 \end{aligned}$$

and $f^{(n)}(2) = 0$ for $n \geq 4$.

$$\begin{aligned} x^3 &= 8 + \frac{12(x-2)}{1!} + \frac{12(x-2)^2}{2!} + \frac{6(x-2)^3}{3!} \\ &= 8 + 12(x-2) + 6(x-2)^2 + (x-2)^3 \end{aligned}$$

13. If $\sum_{n=1}^{100} \frac{1}{n^4}$ is used to approximate $\sum_{n=1}^{\infty} \frac{1}{n^4}$, find an upper bound on the error using the integral test.

Looking at the graph of $\frac{1}{x^4}$, we get that the error, $\sum_{n=101}^{\infty} \frac{1}{n^4}$ is at most

$$\begin{aligned} \int_{100}^{\infty} \frac{1}{x^4} dx = \int_{100}^{\infty} x^{-4}\, dx &= \left[-\frac{x^{-3}}{3} \right]_{100}^{\infty} \\ &= \frac{10^{-6}}{3}. \end{aligned}$$

14. The Maclaurin series of $\frac{1}{1+x^3}$ is $\sum_{n=0}^{\infty} (-1)^n x^{3n}$. Use it to write $\int_0^{\frac{1}{2}} \frac{1}{1+x^3} dx$ as an infinite series and estimate the error if the first three non-zero terms are used to approximate $\int_0^{\frac{1}{2}} \frac{1}{1+x^3} dx$.

$$\begin{aligned} \int_0^{\frac{1}{2}} \frac{1}{1+x^3} dx &= \sum_{n=0}^{\infty} (-1)^n \int_0^{\frac{1}{2}} x^{3n}\, dx \\ &= \sum_{n=0}^{\infty} (-1)^n \left[\frac{x^{3n+1}}{3n+1} \right]_0^{1/2} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(3n+1)2^{3n+1}}. \end{aligned}$$

The error is $\sum_{n-3}^{\infty} (-1)^n \frac{1}{(3n+1)2^{3n+1}}$ so, by the alternating series estimate, the error is at most $\frac{1}{10 \cdot 2^{10}}$.

15. Find the interval of convergence of $\sum_{n=1}^{\infty} \frac{n^2}{2^n}(x-3)^n$, remembering to check the endpoints. Name any tests that you use.

The ratio test gives

$$\lim_{n\to\infty} \left| \frac{(n+1)^2(x-3)^{n+1}2^n}{2^{n+1}n^2(x-3)^n} \right| = \lim_{n\to\infty} \frac{|x-3|}{2},$$

so we have convergence if $|x-3| < 2$, or x in (1,5). At $x = 1$ the series is $\Sigma \frac{n^2}{2^n}(-2)^n = \Sigma(-1)^n n^2$ which diverges since $a_n \not\to 0$. At $x = 5$ we have $\Sigma \frac{n^2 2^n}{2^n} = \Sigma n^2$ which diverges, again since $a_n \not\to 0$. The interval of convergence is then (1,5).