Fall 2004 Math 151
2 Limits and Rates of Change
2.1 Tangent and Velocity Problems
Mon, 06/Sep ©2004, Art Belmonte

Summary

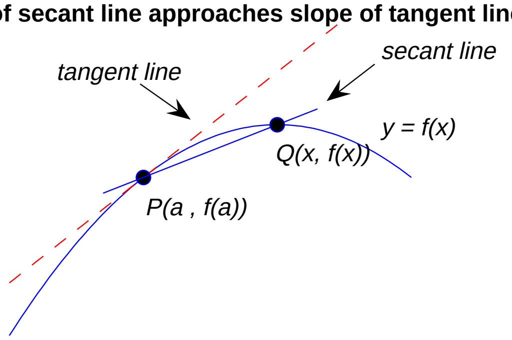
In this section, we deduce velocities and slopes of tangent lines by looking at evidence in numerical tables generated via MATLAB.

That said, we shall appeal to the formal definition (in terms of limits) from Section 2.7 to summarize the essential concepts in a concise fashion. *Do not be alarmed!* This summary will be fleshed out with concrete examples in this lecture.

We'll have ample opportunity in sections 2.2–2.6 to work with limits analytically and absorb their more precise formulation. For now, however, let's just think of limits as real numbers or vectors that various numerical expressions appear to approach on the basis of the empirical evidence we see in tables.

- The **secant line** to the curve $y = f(x)$ at the point $P(a, f(a))$ through the nearby distinct point $Q(x, f(x))$ is the line through P with slope $m_{\text{sec}} = \dfrac{f(x) - f(a)}{x - a}$. Geometrically, the secant line at P is a "rough" straight-line approximation to the curve f near P. The closer Q is to P, the better the secant line approximates the curve f at P —at least close by.

Slope of secant line approaches slope of tangent line.



- The **tangent line** to the curve $y = f(x)$ at $P(a, f(a))$ is the line through P with slope $m_{\tan} = \lim_{x\to a} \dfrac{f(x) - f(a)}{x - a}$, provided that this limit exists; i.e., provided [in this lecture] that the numerical slopes in a table approach a definitive real number $m_{\tan}$ as x approaches a. Geometrically, the tangent line at P is the "best" straight-line approximation to the curve f near P (in a manner to be made precise in later sections).

- Let C be the graph of a vector function $\mathbf{r}(t) = [x(t), y(t)]$. The **secant vector** from the point $\mathbf{P} = \mathbf{r}(a)$ [identified as a position vector] through the nearby distinct point $\mathbf{Q} = \mathbf{r}(t)$ is given by $\mathbf{v}_{\text{sec}} = \dfrac{1}{t-a}(\mathbf{r}(t) - \mathbf{r}(a))$; equivalently (letting $h = t - a$), we have $\mathbf{v}_{\text{sec}} = \dfrac{1}{h}(\mathbf{r}(a+h) - \mathbf{r}(a))$. Regarding the parameter t as time and $\mathbf{r}(t)$ as a position vector, the secant vector is the **average velocity** vector.

- Moreover, the **tangent vector** to the curve $C \leftrightarrow \mathbf{r}(t)$ at $\mathbf{P}$ is

$$\mathbf{v} = \mathbf{v}_{\tan} = \lim_{t\to a} \frac{1}{t-a}(\mathbf{r}(t) - \mathbf{r}(a)) = \lim_{h\to 0} \frac{1}{h}(\mathbf{r}(a+h) - \mathbf{r}(a)),$$

  provided this limit exists; i.e., provided [in this lecture] that the numerical vectors in a table approach a definitive vector $\mathbf{v} = \mathbf{v}_{\tan}$ as t approaches a or h approaches 0. Regarding the parameter t as time and $\mathbf{r}(t)$ as a position vector, the tangent vector is the **instantaneous velocity** vector.

- In the event that the tangent vector $\mathbf{v}$ *does* exist, then the **tangent line** to C at $\mathbf{P}$ is given by $\mathbf{L}(t) = \mathbf{r}(a) + t\mathbf{v}$.

MATLAB Examples

s078x06

The location at time t of an object moving in the xy-plane is given by $\mathbf{r}(t) = \left[t^2, t^3 + 1\right]$, where distance is measured in meters and time in seconds. So at time $t = 1$, the object is located at $(1, 2)$ since $\mathbf{r}(1)$ is the position vector $[1, 2]$.

(a) Find the average velocity of the object for the time period that begins at $t = 1$ and lasts h seconds, where h takes on the values 0.5, 0.1, 0.05, 0.01 seconds.

(b) Deduce the instantaneous velocity of the object when $t = 1$ from the average velocities in (a).

Solution

Looking at average velocities, we deduce that the instantaneous velocity is $\mathbf{v} = \mathbf{v}_{\tan} = [2, 3]$. Components are in m/s.

```
%-----------------------------------------------
% Stewart 78/6
%
for h = [0.5 0.1 0.05 0.01]
    v_sec = (r(1+h) - r(1)) / h
end
v_sec =
    2.5000    4.7500
v_sec =
    2.1000    3.3100
v_sec =
    2.0500    3.1525
v_sec =
    2.0100    3.0301
v_tan = [2 3];
%
echo off; diary off
%--------------------
function p = r(t)
p = [t^2, t^3 + 1];
```

REMARKS

- Again, as $h \to 0$, we have $\mathbf{v}_{\text{sec}} \to \mathbf{v}_{\text{tan}} = [2, 3]$ from the numerical evidence.
- The **for-end** loop repeats a sequence of commands a predetermined number of times. Refer to Subsection 7.4.1 of Gilat (your Math 151 MATLAB lab manual).
- The vector function $\mathbf{r}(t)$ is defined as a **function M-file**. See Chapter 6 in Gilat.

s078x07

The point $P(4, 2)$ lies on the curve $y = f(x) = \sqrt{x}$.

(a) If Q is the point$\left(x, \sqrt{x}\right)$, find the slope of the secant line containing P and Q for the following values of x: 3, 3.5, 3.9, 3.99, 3.999, 4.001, 4.01, 4.1, 4.5, 5.

(b) Deduce the slope of the tangent line to the curve at $P(4, 2)$ from the slopes of nearby secant lines in (a).

(c) Find an equation of the tangent line to the curve at $P(4, 2)$.

Solution

It appears that the slope of the tangent line at P is $m_{\text{tan}} = 0.25$ or $\frac{1}{4}$. Using the point-slope formula from high school, an equation of the tangent line at P is $y - 2 = \frac{1}{4}(x - 4)$ or $y = \frac{1}{4}x + 1$.

```
%-------------------------------------------------
% Stewart 78/7
%
format long
slopes = [];
x_vals = [3 3.5 3.9 3.99 3.999 4.001 4.01 4.1 4.5 5];
%
for x = x_vals
    m_sec = (f(x) - f(4)) / (x - 4);
    slopes = [slopes m_sec];
end
%
table = [x_vals' slopes']
disp('           x                m_sec')
%
table =
   3.00000000000000   0.26794919243112
   3.50000000000000   0.25834261322606
   3.90000000000000   0.25158234186850
   3.99000000000000   0.25015644561822
   3.99900000000000   0.25001562695340
   4.00100000000000   0.24998437695292
   4.01000000000000   0.24984394500787
   4.10000000000000   0.24845673131659
   4.50000000000000   0.24264068711928
   5.00000000000000   0.23606797749979
           x                m_sec
m_tan = 0.25
m_tan =
   0.25000000000000
format short
%
echo off; diary off
%--------------------
function y = f(x)
y = sqrt(x);
```

REMARKS

- Note that as $x \to 4$, we have $m_{\text{sec}} \to m_{\text{tan}} = 0.25 = \frac{1}{4}$ from the numerical evidence.
- With each pass through the **for-end** loop, we add another slope to the row vector of slopes of secant lines. (Initially we start with an empty list.)
- The transpose operator (') changes a row vector into a column vector, or *vice versa*. See Section 2.4 in Gilat.

s079x12

The position s of a car (in feet from an origin) at time t (in seconds) is given empirically by the following table.

t	0	1	2	3	4	5
s	0	10	32	70	119	178

(a) Find the average velocity of the car for the time period that begins at $t = 2$ and lasts for h seconds, where h takes on the values 1, 2, 3 seconds. (We'll also compute the case where $h = -1$.)

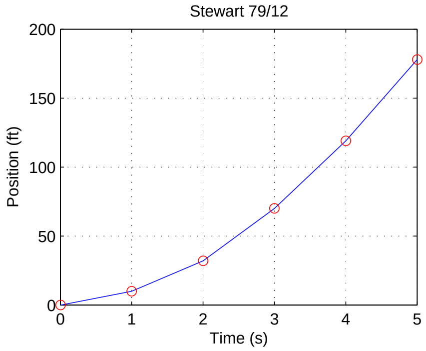
(b) Use the average velocities in (a) as well as the graph of s as a function of t to estimate the instantaneous velocity of the car when $t = 2$.

Solution

We'll average the two average velocities corresponding to the cases where $h = \pm 1$. Accordingly, we estimate the instantaneous velocity of the car at $t = 2$ to be $(22 + 38)/2 = 30$ ft/s. Here is the script M-file (input file driver), followed by the output diary file plus a graph of s versus t.

```
%-------------------------------------------------
% Stewart 79/12
%
t = 0:5
s = [0 10 32 70 119 178]
for h = [-1 1 2 3]
    avg_vel = (s(3+h) - s(3)) / h
end
plot(t,s, t,s,'ro'); grid on
xlabel('Time (s)'); ylabel('Position (ft)')
title('Stewart 79/12')
echo off; diary off
%--------------------
t =
     0     1     2     3     4     5
s =
     0    10    32    70   119   178
avg_vel =
    22
avg_vel =
    38
avg_vel =
   43.5000
avg_vel =

   48.6667
```



REMARKS

- Note how the colon operator (:) generates a row vector of equally spaced points. Refer to Chapter 2 in Gilat or type "**help colon**" (without the quotes) in the Command Window in MATLAB.
- Whereas the ratio $\frac{s(1+h)-s(1)}{h}$ is called the *forward difference quotient*, the ratio $\frac{s(1)-s(1-h)}{h}$ is called the *backward difference quotient*. When we average these two ratios, we obtain the *centered difference quotient*

$$\frac{s(1+h)-s(1-h)}{2h}.$$

  This gives a more accurate estimate of the slope than either the forward or backward difference quotients.