Fall 2004 Math 151
2 Limits and Rates of Change
2.2 The Limit of a Function
Wed, 08/Sep ©2004, Art Belmonte

Summary

In this section, we deduce limits by once again looking at evidence in numerical tables generated via MATLAB. We'll also look at graphs to ascertain certain limits. In later sections we'll use algebraic manipulation together with limit laws to more formally calculate limits. For now we'll state things colloquially.

- **Limit of a scalar function**: We write $\lim_{x\to a} f(x) = L$ and say "the limit of $f(x)$ as x approaches a equals L" if and only if we can make the values of $f(x)$ arbitrarily close to L by choosing x to be sufficiently close to a. (For technical reasons that will become evident during the investigation of limits of difference quotients later in the chapter, we do *not* allow x to equal a.) We also write $f(x) \to L$ as $x \to a$ and say "$f(x)$ approaches L as x approaches a."
- **Left-hand limit**: $\lim_{x\to a^-} f(x) = L$; same idea as the preceding, with the additional provision that $x < a$; i.e., x approaches a through values strictly *less* than a.
- **Right-hand limit**: $\lim_{x\to a^+} f(x) = L$; same idea as the preceding, with the alternative provision that $x > a$; i.e., x approaches a through values strictly *greater* than a.
- **Positive infinite limit**: $\lim_{x\to a} f(x) = \infty$ if and only if the values of $f(x)$ can be made arbitrarily positively large by taking x sufficiently close to a (but again, not equal to a). We also write $f(x) \to \infty$ as $x \to a$ and say "$f(x)$ approaches (or tends to) infinity as x approaches a." Note that positive infinity (∞) is *not* a real number. The aforementioned limit symbolism is just a shorthand description of behavior.
- **Negative infinite limit**: $\lim_{x\to a} f(x) = -\infty$ if and only if the values of $f(x)$ can be made arbitrarily negatively large by taking x sufficiently close to a (but again, not equal to a). We also write $f(x) \to -\infty$ as $x \to a$ and say "$f(x)$ approaches (or tends to) negative infinity as x approaches a." Negative infinity ($-\infty$) is *not* a real number. The aforementioned limit symbolism is just a shorthand description of behavior.
- NOTE: Analogous one-sided infinite limits may also be considered: $\lim_{x\to a^+} f(x) = \infty$, $\lim_{x\to a^-} f(x) = \infty$, $\lim_{x\to a^+} f(x) = -\infty$, and $\lim_{x\to a^-} f(x) = -\infty$.
- **Vertical asymptote**: The vertical line $x = a$ provided one of the six aforementioned infinite limits is manifested.
- **Limit of a vector function**: We write $\lim_{t\to a} \mathbf{r}(t) = \mathbf{b}$ and say "the limit of $\mathbf{r}(t)$ as t approaches a equals $\mathbf{b}$" if and only if we can make the vector values $\mathbf{r}(t)$ arbitrarily close to $\mathbf{b}$ by choosing t to be sufficiently close to a, without letting t equal a (for technical reasons.) We also write $\mathbf{r}(t) \to \mathbf{b}$ as $t \to a$ and say "$\mathbf{r}(t)$ approaches $\mathbf{b}$ as t approaches a." In practice, to find the limit of a vector function, simply find the limits of its constituent scalar functions. That is, if $\mathbf{r}(t) = [x(t), y(t)]$, then $\lim_{t\to a} \mathbf{r}(t) = \left[\lim_{t\to a} x(t), \lim_{t\to a} y(t)\right]$. In other words, "the limit of the vector is the vector of the limits."

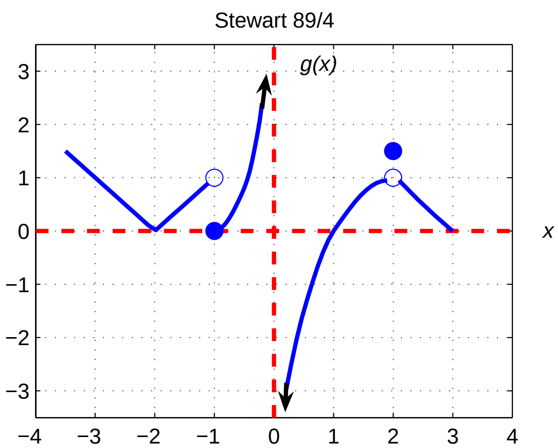
Hand Examples

89/4

State the value of the limit from the graph, if it exists. Otherwise, state that the limit does not exist (DNE).

(a) $\lim_{x\to 1} g(x)$ (b) $\lim_{x\to 0} g(x)$ (c) $\lim_{x\to 2} g(x)$

(d) $\lim_{x\to -2} g(x)$ (e) $\lim_{x\to -1^-} g(x)$ (f) $\lim_{x\to -1} g(x)$



Solution

(a) $\lim_{x\to 1} g(x) = 0$ (b) $\lim_{x\to 0} g(x)$ DNE (c) $\lim_{x\to 2} g(x) = 1$

(d) $\lim_{x\to -2} g(x) = 0$ (e) $\lim_{x\to -1^-} g(x) = 1$ (f) $\lim_{x\to -1} g(x)$ DNE

REMARKS

- In (b), there is no *common* infinite limit. That is, while $g(x) \to \infty$ as x approaches 0 from the left and $g(x) \to -\infty$ as x approaches 0 from the right, the behaviors differ. The sign of $g(x)$ differs depending on the direction of approach.
- In (c), the left-hand and right-hand limits are the same. Accordingly, the two-sided limit *does* exist. The fact that $g(2)$ is not equal to this limiting value is irrelevant. When we take limits as x approaches 2, we never let x actually *attain* the value 2.
- In (f), there is no common limit. Whereas $g(x) \to 1$ as x approaches -1 from the left, we see that $g(x) \to 0$ as x approaches -1 from the right. The one-sided limits differ. Hence the two-sided limit does not exist.

MATLAB Examples

s090x08

(a) Sketch the graph of the function

$$g(x) = \begin{cases} 2 - x & \text{if } x < -1, \\ x & \text{if } -1 \le x < 1, \\ 4 & \text{if } x = 1, \\ 4 - x & \text{if } x > 1. \end{cases}$$

(b) State the value of the limit from the graph, if it exists. Otherwise, state that the limit does not exist (DNE).

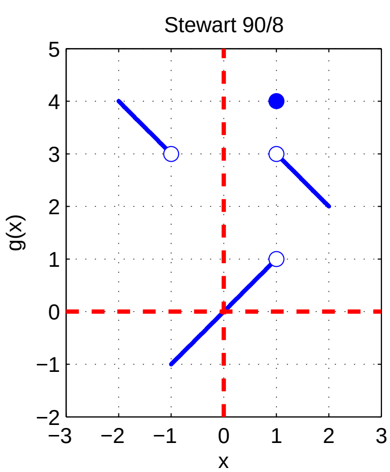
(i) $\lim_{x \to -1^-} g(x)$ (ii) $\lim_{x \to -1^+} g(x)$ (iii) $\lim_{x \to -1} g(x)$

(iv) $\lim_{x \to 1^-} g(x)$ (v) $\lim_{x \to 1^+} g(x)$ (vi) $\lim_{x \to 1} g(x)$

Solution

Piecewise-defined functions are readily implemented in MATLAB using logical operators. Read all about them in Section 7.1 of Gilat.

```
%-------------------------------------------------------
% Stewart 90/8
%
% MAIN ACTION
x = linspace(-2, 2, 401);
g = (x<-1) .* (2-x) + ...
    (-1<=x & x<1) .* x + ...
    (x==1) .* 4 + ...
    (x>1) .* (4-x);
plot(x,g,'.'); grid on
axis equal; axis([-3 3 -2 5])
% EXTRA FLUFF
hold on
plot(1,4,'o', 'MarkerFaceColor', 'b', ...
    'MarkerSize', 6)
plot([-3 3], [0 0], 'r--', 'LineWidth', 2)
plot([0 0], [-2 5], 'r--', 'LineWidth', 2)
xlabel('x'); ylabel('g(x)')
title('Stewart 90/8')
%
echo off; diary off
```



(i) $\lim_{x \to -1^-} g(x) = 3$ (ii) $\lim_{x \to -1^+} g(x) = -1$ (iii) $\lim_{x \to -1} g(x)$ DNE

(iv) $\lim_{x \to 1^-} g(x) = 1$ (v) $\lim_{x \to 1^+} g(x) = 3$ (vi) $\lim_{x \to 1} g(x)$ DNE

s090x10

Evaluate the scalar function $g(x) = \dfrac{1 - x^2}{x^2 + 3x - 10}$ at these x-values: $x = 3, 2.1, 2.01, 2.001, 2.0001, 2.00001$. Use the function values $g(x)$ to guess the value of the limit $\lim_{x \to 2^+} g(x)$, or state that it does not exist.

Solution

From the numerical evidence in the table below, we see that $\lim_{x \to 2^+} g(x) = -\infty$. [Indeed, $g(2.00001) \approx -4.3 \times 10^4$.]

```
%-------------------------------------------------------
% Stewart 90/10
%
format long
x = [3 2.1 2.01 2.001 2.0001 2.00001];
g = eval(vectorize('(1 - x^2) / (x^2 + 3*x - 10)'));
% Here are the x- and g-values.
table = [x' g']
table =
   1.0e+04 *
   0.00030000000000  -0.00010000000000
   0.00021000000000  -0.00048028169014
   0.00020100000000  -0.00433680456491
   0.00020010000000  -0.04290817026140
   0.00020001000000  -0.42862244967761
   0.00020000100000  -4.28576530613665
format short
%
echo off; diary off
```

s090x15

Evaluate the vector function $\mathbf{r}(t) = \left[2t - 1, \dfrac{t^2 - 1}{t - 1} \right]$ at t-values $t = 0.2, 0.4, 0.6, 0.8, 0.9, 0.99, 1.01, 1.1, 1.2, 1.4, 1.6, 1.8$. Use the vector function values $\mathbf{r}(t)$ to guess the value of the limit $\lim_{t \to 1} \mathbf{r}(t)$, or state that it does not exist.

Solution

It appears as if $\lim_{t \to 1} \mathbf{r}(t) = [1, 2]$.

Solution

Here are the script M-file driver, the vector function M-file, and the diary output.

```
%-------------------------------------------------------
% Stewart 90/15
%
t_vals = [0.2 0.4 0.6 0.8 0.9 0.99 ...
    1.01 1.1 1.2 1.4 1.6 1.8];
w = [];
for t = t_vals
    w = [w; r(t)];
end
table = [t_vals' w]
disp('       t       [ r(1)      r(2) ]')
%
echo off; diary off
%--------------------
function w = r(t)
w = [2*t - 1, (t^2 - 1) / (t-1)];
```

```
%--------------------
table =
    0.2000   -0.6000    1.2000
    0.4000   -0.2000    1.4000
    0.6000    0.2000    1.6000
    0.8000    0.6000    1.8000
    0.9000    0.8000    1.9000
    0.9900    0.9800    1.9900
    1.0100    1.0200    2.0100
    1.1000    1.2000    2.1000
    1.2000    1.4000    2.2000
    1.4000    1.8000    2.4000
    1.6000    2.2000    2.6000
    1.8000    2.6000    2.8000

       t      [ r(1)     r(2) ]
```

s090x30

Estimate the value of $\lim_{x \to 0} \dfrac{6^x - 2^x}{x}$ by graphing the function $y = \dfrac{6^x - 2^x}{x}$ near $x = 0$. State your answer to two decimal places.

Solution

It appears as if $\lim_{x \to 0} \dfrac{6^x - 2^x}{x} \approx 1.0985$. (The exact value is $\ln 3$, as we'll see later in the course.)

Solution

```
%--------------------------------------------------
% Stewart 91/30
%
x = linspace(-0.001, 0.001, 101);
y = (6.^x - 2.^x) ./ x;
Warning: Divide by zero.
plot(x,y); grid on
xlabel('x'); ylabel('y')
title('Stewart 91/30')
%
echo off; diary off
```