

Fall 2004 Math 151

2 Limits and Rates of Change

2.6 Limits at Infinity;

Horizontal Asymptotes

Wed, 15/Sep ©2004, Art Belmonte

Summary

- **Limits at infinity:** Let f be a function defined on $[a, \infty)$. We write $\lim_{x \rightarrow \infty} f(x) = L$ (or alternatively $f(x) \rightarrow L$ as $x \rightarrow \infty$) to signify that values of f can be made as close to L as we please by taking sufficiently large positive values of x . Similarly, $\lim_{x \rightarrow -\infty} f(x) = L$ (or alternatively $f(x) \rightarrow L$ as $x \rightarrow -\infty$) signifies that values of f can be made as close to L as we please by taking sufficiently large negative values of x . In either case, the horizontal line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$. Most of the limit laws discussed in Section 2.3 apply to limits at infinity. For example, suppose that $f(x) \rightarrow L$ and $g(x) \rightarrow M$ as $x \rightarrow \infty$. Then we have $f(x) + g(x) \rightarrow L + M$ as $x \rightarrow \infty$.

- **reciprocal power limits at infinity:** Two useful limits are $\lim_{x \rightarrow \infty} \frac{1}{x^r} = 0$ for any positive rational number r and $\lim_{x \rightarrow -\infty} \frac{1}{x^q} = 0$ for any positive rational number q such that x^q is defined for all negative values of x .

- **infinite limits at infinity:** The symbolism $\lim_{x \rightarrow \infty} f(x) = \infty$ means that the function values $f(x)$ become arbitrarily large once x becomes sufficiently positively large. In the same fashion, the following have the analogous connotations:
 $\lim_{x \rightarrow \infty} f(x) = -\infty$, $\lim_{x \rightarrow -\infty} f(x) = \infty$, $\lim_{x \rightarrow -\infty} f(x) = -\infty$.

Hand Examples

133/4

Evaluate the limit $\lim_{t \rightarrow \infty} \frac{7t^3 + 4t}{2t^3 - t^2 + 3}$. Justify steps by citing the appropriate properties of limits.

Solution

Start by dividing the numerator and denominator of the rational limiting expression by the highest power of t contained therein;

in this instance, t^3 . Then apply limit properties.

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{7t^3 + 4t}{2t^3 - t^2 + 3} &= \lim_{t \rightarrow \infty} \left(\frac{7 + \frac{4}{t^2}}{2 - \frac{1}{t} + \frac{3}{t^2}} \right) \\ &= \frac{\lim_{t \rightarrow \infty} \left(7 + \frac{4}{t^2} \right)}{\lim_{t \rightarrow \infty} \left(2 - \frac{1}{t} + \frac{3}{t^2} \right)} \quad [\text{quotient law}] \\ &= \frac{\lim_{t \rightarrow \infty} 7 + \lim_{t \rightarrow \infty} \frac{4}{t^2}}{\lim_{t \rightarrow \infty} 2 - \lim_{t \rightarrow \infty} \frac{1}{t} + \lim_{t \rightarrow \infty} \frac{3}{t^2}} \quad [\text{sum law}] \\ &= \frac{7 + 0}{2 - 0 + 0} = \frac{7}{2} \end{aligned}$$

133/12

Find the limit $\lim_{x \rightarrow -\infty} \frac{\sqrt{x^2 + 4x}}{4x + 1}$.

Solution

First off, for $x < 0$ we have $\sqrt{x^2} = |x| = -x$. Now proceed with the algebraic manipulations.

$$\begin{aligned} \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2 + 4x}}{4x + 1} &= \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2} \sqrt{1 + \frac{4}{x}}}{4x + 1} \\ &= \lim_{x \rightarrow -\infty} \frac{-x \sqrt{1 + \frac{4}{x}}}{x \left(4 + \frac{1}{x} \right)} \\ &= \lim_{x \rightarrow -\infty} \frac{-\sqrt{1 + \frac{4}{x}}}{4 + \frac{1}{x}} \\ &= \frac{-\sqrt{1 + 0}}{4 + 0} = -\frac{1}{4} \end{aligned}$$

133/14

Find the limit $\lim_{x \rightarrow \infty} \left(\sqrt{x^2 + 3x + 1} - x \right)$.

Solution

Let's see, $\infty - \infty = 0$, right? NO: $\infty - \infty$ is meaningless! Instead, rationalize the numerator (the denominator here is 1). Also, for $x > 0$ we have $\sqrt{x^2} = |x| = x$.

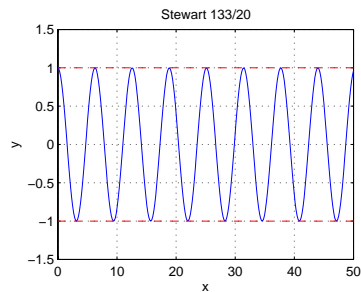
$$\begin{aligned}
& \lim_{x \rightarrow \infty} \left(\frac{\sqrt{x^2 + 3x + 1} - x}{1} \cdot \frac{\sqrt{x^2 + 3x + 1} + x}{\sqrt{x^2 + 3x + 1} + x} \right) \\
&= \lim_{x \rightarrow \infty} \frac{(x^2 + 3x + 1) - x^2}{\sqrt{x^2 + 3x + 1} + x} \\
&= \lim_{x \rightarrow \infty} \frac{3x + 1}{\sqrt{x^2 + 3x + 1} + x} \\
&= \lim_{x \rightarrow \infty} \frac{x \left(3 + \frac{1}{x} \right)}{x \left(\sqrt{1 + \frac{3}{x} + \frac{1}{x^2}} + 1 \right)} \quad [\text{Factor out } x = \sqrt{x^2}.] \\
&= \lim_{x \rightarrow \infty} \frac{3 + \frac{1}{x}}{\sqrt{1 + \frac{3}{x} + \frac{1}{x^2}} + 1} = \frac{3}{2}
\end{aligned}$$

133/20

Find $\lim_{x \rightarrow \infty} \cos x$.

Solution

The limit does not exist. The values of the cosine function continually vary between all real numbers in the interval $[-1, 1]$. The limit *diverges by oscillation*, as one look at a graph will show.



MATLAB Examples

s26eA [Section 2.6 / Example A]

Find all horizontal and vertical asymptotes of

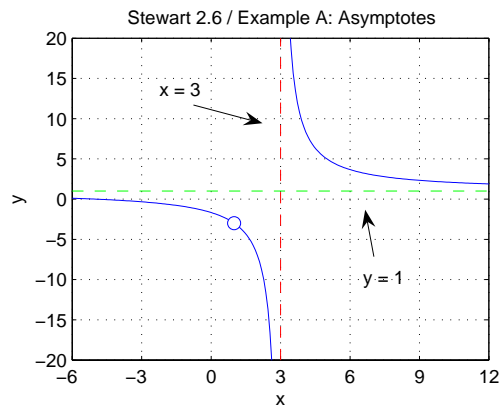
$$f(x) = \frac{x^2 + 4x - 5}{x^2 - 4x + 3}$$

Sketch a graph of this rational function and its asymptotes.

Solution

- Since $\lim_{x \rightarrow \infty} \frac{x^2 + 4x - 5}{x^2 - 4x + 3} = \lim_{x \rightarrow \infty} \frac{1 + \frac{4}{x} - \frac{5}{x^2}}{1 - \frac{4}{x} + \frac{3}{x^2}} = 1$, we see that $y = 1$ is a horizontal asymptote.
- Also, $\lim_{x \rightarrow -\infty} \frac{x^2 + 4x - 5}{x^2 - 4x + 3} = 1$ reveals this same asymptote.
- A vertical asymptote *may* occur where the denominator equals zero. (We still need to check the relevant limit involved!) Now $x^2 - 4x + 3 = (x - 1)(x - 3)$ equals zero if $x = 1$ or $x = 3$.
- Since $\lim_{x \rightarrow 1} \frac{x^2 + 4x - 5}{x^2 - 4x + 3} = \lim_{x \rightarrow 1} \frac{(x - 1)(x + 5)}{(x - 1)(x - 3)} = \lim_{x \rightarrow 1} \frac{x + 5}{x - 3} = -3$, we see that $x = 1$ is *not* a vertical asymptote of f .
- However, $\lim_{x \rightarrow 3^+} \frac{x^2 + 4x - 5}{x^2 - 4x + 3} = \lim_{x \rightarrow 3^+} \frac{(x - 1)(x + 5)}{(x - 1)(x - 3)} = \lim_{x \rightarrow 3^+} \frac{x + 5}{x - 3} = \infty$ shows that f has a vertical asymptote at $x = 3$.

A picture will make all this clear. The code that produced it follows.



```

%-----
% Stewart 2.6 / Example A
%
% Function as string
f = '(x^2 + 4*x - 5) / (x^2 - 4*x + 3)';
% Left piece
x = linspace(-6, 3);
y = eval(vectorize(f));
Warning: Divide by zero.
plot(x,y); grid on; hold on
% Right piece
x = linspace(3, 12);
y = eval(vectorize(f));
Warning: Divide by zero.
plot(x,y)
% Vertical asymptote
plot([3 3], [-20 20], 'r--')
% Horizontal asymptote
plot([-6, 12], [1 1], 'g--')
%
axis([-6 12 -20 20])

```

```

xlabel('x'); ylabel('y')
title('Stewart 2.6 / Example A: Asymptotes')
set(gca, 'Xtick', [-6 : 3 : 12])
set(gca, 'Ytick', [-20 : 5 : 20])
plot(1, -3, 'o', 'MarkerFaceColor', 'w', ...
     'MarkerSize', 7)
%
echo off; diary off

```

s26eB [Section 2.6 / Example B]

Find all horizontal, vertical, and slanted asymptotes of

$$g(x) = \frac{x^2}{2x + 5}.$$

Solution

- Now $\lim_{x \rightarrow \infty} \frac{x^2}{2x + 5} = \lim_{x \rightarrow \infty} \frac{x}{2 + \frac{5}{x}} = \infty$. Similarly, $\lim_{x \rightarrow -\infty} \frac{x^2}{2x + 5} = \lim_{x \rightarrow -\infty} \frac{x}{2 + \frac{5}{x}} = -\infty$. Accordingly, there are no horizontal asymptotes of g .

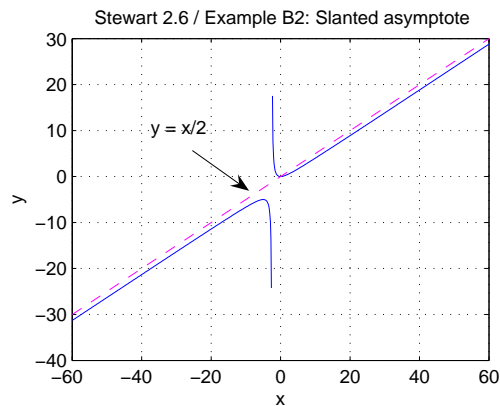
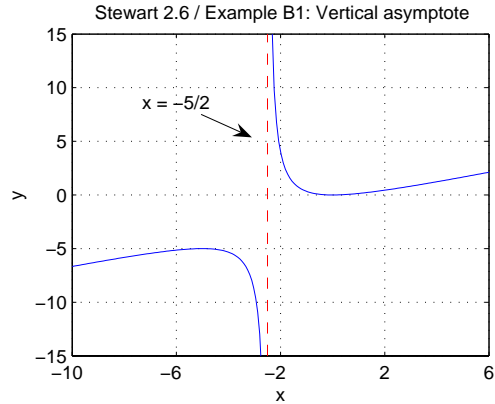
- Since $\lim_{x \rightarrow (-5/2)^+} \frac{x^2}{2x + 5} = \infty$, we see that $x = -\frac{5}{2}$ is a vertical asymptote of g .

- In the first bullet, we see that as $x \rightarrow \infty$ the values of $\frac{x^2}{2x + 5} = \frac{x}{2 + \frac{5}{x}}$ get closer and closer to those of the straight line $y = \frac{x}{2}$; we write, $\frac{x^2}{2x + 5} \sim \frac{x}{2}$. More formally, if $\lim_{x \rightarrow \infty} \frac{f(x)}{L(x)} = 1$ where $L(x) = mx + b$, then L is a slanted asymptote of f . Therefore,

$$\lim_{x \rightarrow \infty} \frac{\frac{x^2}{2x + 5}}{\frac{x}{2}} = \lim_{x \rightarrow \infty} \frac{2x^2}{2x^2 + 5x} = \lim_{x \rightarrow \infty} \frac{2}{2 + \frac{5}{x}} = 1$$

shows that $y = \frac{x}{2}$ is a slanted asymptote of g .

- Similarly, as $x \rightarrow -\infty$, we have $\frac{x^2}{2x + 5} \sim \frac{x}{2}$, again showing that $y = \frac{x}{2}$ is a slanted asymptote.



```

-----
% Stewart 2.6 / Example B1: Vertical asymptote
%
% Function as string
f = 'x^2 / (2*x + 5)';
% Left piece
x = linspace(-10, -2.5);
y = eval(vectorize(f));
Warning: Divide by zero.
plot(x,y); grid on; hold on
% Right piece
x = linspace(-2.5, 6);
y = eval(vectorize(f));
Warning: Divide by zero.
plot(x,y)
% Vertical asymptote
plot([-2.5 -2.5], [-15 15], 'r--')
%
axis([-10 6 -15 15])
xlabel('x'); ylabel('y')
title('Stewart 2.6 / Example B1: Vertical asymptote')
set(gca, 'Xtick', [-10 : 4 : 6])
set(gca, 'Ytick', [-15 : 5 : 15])
%
echo off; diary off
-----
% Stewart 2.6 / Example B2: Slanted asymptote
%
% Function as string
f = 'x^2 / (2*x + 5)';
% Left piece
x = linspace(-60, -2.5, 400);
y = eval(vectorize(f));
Warning: Divide by zero.
plot(x,y); grid on; hold on
% Right piece
x = linspace(-2.5, 60, 400);
y = eval(vectorize(f));
Warning: Divide by zero.
plot(x,y)
% Slanted asymptote

```

```
x = linspace(-60, 60, 10);
y = x/2;
plot(x,y, 'm--')
%
xlabel('x'); ylabel('y')
title('Stewart 2.6 / Example B2: Slanted asymptote')
%
echo off; diary off
```

s133x32

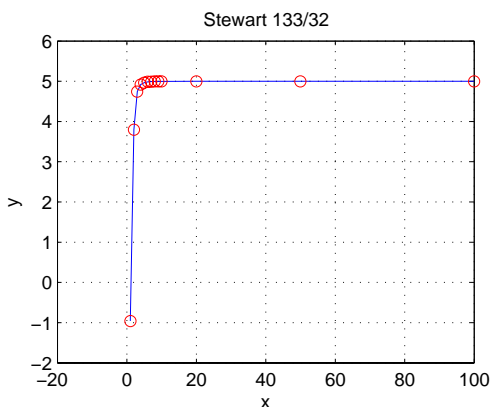
Guess the value of the limit $\lim_{x \rightarrow \infty} \left(x^2 \sin \left(\frac{5}{x^2} \right) \right)$ by evaluating

$x^2 \sin \left(\frac{5}{x^2} \right)$ for $x = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 50, 100$.

Then confirm your guess by evaluating this limit exactly.

Solution

It appears that the limit is 5. MATLAB' **limit** command corroborates this.



```
%-----
% Stewart 133/32
%
f = 'x^2 * sin(5 / x^2)';
x = [1:10 20 50 100];
y = eval(vectorize(f));
plot(x,y, x,y,'ro'); grid on
axis([-20 100 -2, 6])
xlabel('x'); ylabel('y')
title('Stewart 133/32')
%
syms x
f = sym(f);
pretty(f)

%
val = limit(f, x, inf)

val =

5

%
echo off; diary off
```

$$x^2 \sin\left(\frac{5}{x^2}\right)$$

Infinite limits via the *limit* command

Here are transcripts of the first three hand exercises above dispatched via the **limit** command. Infinity (∞) is specified as **inf** and $-\infty$ as **-inf**.

```
%-----
% Stewart 133/4
%
syms t
f = (7*t^3 + 4*t) / (2*t^3 - t^2 + 3);
pretty(f)

%
val = limit(f, t, inf)

val =

7/2

%
echo off; diary off
%-----
% Stewart 133/12
%
syms x
f = sqrt(x^2 + 4*x) / (4*x + 1);
pretty(f)

%
val = limit(f, x, -inf)

val =

-1/4

%
echo off; diary off
%-----
% Stewart 133/14
%
syms x
f = sqrt(x^2 + 3*x + 1) - x;
pretty(f)

%
val = limit(f, x, inf)

val =

3/2

%
echo off; diary off
```

$$\frac{7t^3 + 4t}{2t^3 - t^2 + 3}$$

$$\frac{(x^2 + 4x)^{1/2}}{4x + 1}$$

$$(x^2 + 3x + 1)^{1/2} - x$$