Fall 2004 Math 151
3 Derivatives
3.5 The Chain Rule
Wed, 29/Sep ©2004, Art Belmonte

Summary

In the following, the requisite derivatives are assumed to exist.

- **Chain Rule**: Let h be the composition of f and g; that is, $h = f \circ g$ or $h(x) = f(g(x))$. Then $h'(x) = f'(g(x))g'(x)$. Equivalently, let $y = f(u)$ and $u = g(x)$. Then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

- **General Power Rule**: Let r be any real number and $u = g(x)$ be differentiable. Then $\frac{d}{dx}\left(u^r\right) = ru^{r-1}\frac{du}{dx}$.

Hand Examples

192/2

Find $\frac{dy}{dx}$ and $\left.\frac{dy}{dx}\right|_{x=1}$ two ways, with and without the Chain Rule. Here $y = u^2 - 2u + 3$ and in turn $u = 5 - 6x$.

Solution

- The Chain Rule gives $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (2u-2)(-6) = m$. When $x = 1$, $u = -1$, we have $m = 24$.
- Substituing $u = 5 - 6x$ into y and expanding gives

  $y = (5-6x)^2 - 2(5-6x) + 3 = 25 - 60x + 36x^2 - 10 + 12x + 3$

  or $y = 36x^2 - 48x + 18$. Hence $dy/dx = 72x - 48 = 24$ when $x = 1$.

As they said in the Empire, "All roads lead to Rome!" That said, the Chain Rule is easier since we're working with smaller pieces.

192/8

Find the derivative of $g(t) = \left(6t^2+5\right)^3\left(t^3-7\right)^4$.

Solution

Use the product and chain rules as needed.

$$g'(t) = 3\left(6t^2+5\right)^2(12t)\left(t^3-7\right)^4 + \left(6t^2+5\right)^3 \cdot 4\left(t^3-7\right)^3\left(3t^2\right)$$

Now leave well enough alone. In this form, it is easier for the grader to check our work. If desired, clean it up a wee bit.

$$g'(t) = 36t\left(6t^2+5\right)^2\left(t^3-7\right)^4 + 12t^2\left(6t^2+5\right)^3\left(t^3-7\right)^3$$

Do not fully expand the expression. It's a world 'o' hurt!

```
%--------------------------------------------------
% Stewart 192/8
%
syms t
p = [6 0 5]; q = [1 0 0 -7];
a = conv(p,p); P = conv(a,p); % p^3
b = conv(q,q); Q = conv(b,b); % q^4
g = conv(P,Q); % p^3 * q^4
dg_dt = polyder(g); % derivative of g: coefficients
gp = poly2sym(dg_dt, t); % derivative of g: symbolic
pretty(gp)

      17        15          14        13           12           11
3888 t  + 8640 t   - 90720 t  + 6300 t   - 196560 t   + 763548 t
        10           9          8          7          6
- 138600 t   + 1587600 t  - 2698668 t  + 1058400 t  - 5186160 t
         5           4          3          2
+ 3332196 t  - 3087000 t  + 5186160 t  - 514500 t  + 2160900 t
%
echo off; diary off
```

192/15

Let $F(y) = \left(\frac{y-6}{y+7}\right)^3$. Find $F'(y)$.

Solution

We have

$$\begin{aligned} F'(y) &= 3\left(\frac{y-6}{y+7}\right)^2 \frac{(y+7)(1)-(y-6)(1)}{(y+7)^2} \\ &= \frac{39(y-6)^2}{(y+7)^4}. \end{aligned}$$

192/23

Find the derivative of $y = \cos\left(x^3\right)$.

Solution

We have $y' = -\sin\left(x^3\right)\cdot 3x^2 = -3x^2\sin\left(x^3\right)$.

192/24

Find the derivative of $y = \cos^3 x$.

Solution

Now $y = (\cos x)^3$. So $y' = 3(\cos x)^2(-\sin x) = -3\sin x \cos^2 x$.

193/59

Suppose $h(x) = f(g(x))$ with

$g(3)$	$g'(3)$	$f'(3)$	$f'(6)$
6	4	2	7

Find $h'(3)$.

Solution

We have

$$\begin{aligned} h'(x) &= f'(g(x))g'(x) \\ h'(3) &= f'(g(3))g'(3) \\ h'(3) &= f'(6)g'(3) \\ h'(3) &= (7)(4) = 28. \end{aligned}$$

MATLAB Examples

s193x42

Find the derivative of $y = \sin(\sin(\sin x))$.

Solution

We see that MATLAB recursively applies the Chain Rule.

```
%------------------------------------------------
% Stewart 193/42
%
syms x
y = sin(sin(sin(x)));
yp = diff(y,x); pretty(yp)

                    cos(sin(sin(x))) cos(sin(x)) cos(x)
%
echo off; diary off
```

s193x50

Find an equation of the tangent line to the curve $y = \sqrt{x + \frac{1}{x}}$ at the point $\left(1, \sqrt{2}\right)$.

Solution

The (horizontal) tangent line is $y = \sqrt{2}$, as seen below.

```
%------------------------------------------------
% Stewart 193/50
%
syms x
y = sqrt(x + 1/x);
yp = diff(y,x); pretty(yp)

                                        1
                                  1 - ----
                                      2
                                     x
                               1/2 ------------
                                            1/2
                                   (x + 1/x)
m = subs(yp, x, sym(1))

m =

0


TL = sqrt(2) + m*(x - 1); pretty(TL)

                                       1/2
                                      2
s2 = sqrt(2);
%
h = 0.8;
x = linspace(1-h, 1+h);
y = eval(vectorize(y));
plot(x,y, [1-h 1+h],[s2 s2],'r--'); grid on; hold on
legend('function', 'tangent line', ...
    'Location', 'North')
plot(1, sqrt(2), 'go', 'MarkerFaceColor', 'g', ...
    'MarkerSize', 7)
xlabel('x')
title('Stewart 193/50')
axis equal; axis([0 2 1 3])
set(gca, 'Ytick', 1 : 0.5 : 3)
%
echo off; diary off
```