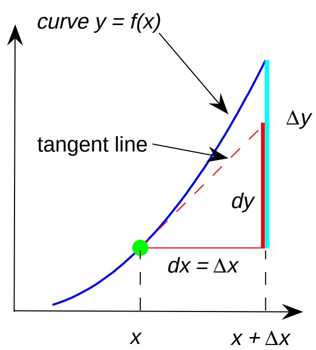
Fall 2004 Math 151
3 Derivatives
3.11 Differentials; Linear and Quadratic Approximations
Wed, 13/Oct ©2004, Art Belmonte

Summary

- Let $y = f(x)$ be a differentiable function. The **differential** dx is an independent variable. The **differential** dy is then defined by the product $dy = f'(x)\,dx$. Its geometrical significance is that $\Delta y \approx dy$. In other words, near a given point on the graph of $y = f(x)$, the actual change in the function along the curve is approximately equal to the change dy along the tangent line.

$$\begin{aligned} \Delta y &\approx dy \\ f(x+dx) - f(x) &\approx dy \\ f(x+dx) &\approx f(x) + dy \\ f(x+dx) &\approx f(x) + f'(x)dx \end{aligned}$$



- **linear approximation** (or tangent line approximation)

$$f(x) \approx f(a) + f'(a)(x-a)$$

- **quadratic approximation**

$$f(x) \approx f(a) + f'(a)(x-a) + \tfrac{1}{2} f''(a)(x-a)^2$$

Hand Examples

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Find the differential dy for the function $y = x \tan x$.

Solution

We have $dy = f'(x)\,dx = \left(\tan x + x \sec^2 x\right) dx$.

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Let $y = \sin x$.

(a) Find the differential dy.

(b) Evaluate dy for $x = \frac{\pi}{6}$ and $dx = -0.1$.

Solution

(a) We have $dy = f'(x)\,dx = \cos x\,dx$.

(b) For $x = \frac{\pi}{6}$ and $dx = -0.1$,
$dy = \left(\cos \frac{\pi}{6}\right)\left(-\frac{1}{10}\right) \approx \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{10}\right) = -\frac{\sqrt{3}}{20} \approx -0.0866$.

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Use differerentials to find to approximate $\sqrt[3]{1.02} + \sqrt[4]{1.02}$.

Solution

- Let $y = f(x) = x^{1/3} + x^{1/4}$. So $f'(x) = \frac{1}{3}x^{-2/3} + \frac{1}{4}x^{-3/4}$.
- Let $x = 1$ and $dx = \Delta x = 1.02 - 1 = +0.02$. Then

$$\begin{aligned} f(x+dx) &\approx f(x) + f'(x)\,dx \\ f(1.02) &\approx f(1) + f'(1)\,dx \\ \sqrt[3]{1.02} + \sqrt[4]{1.02} &\approx 2 + \left(\tfrac{1}{3} + \tfrac{1}{4}\right)(+0.02) \\ &\approx 2.01167 \end{aligned}$$

- Check: MATLAB gives $1.02^{1/3} + 1.02^{1/4} \approx 2.01159$.

MATLAB Examples

s227x18

Let $y = f(x) = x^4 + x^2 + 1$. Compute Δy, dy, and $\Delta y - dy$ for $x = 1$ and $dx = \Delta x = 1, 0.5, 0.1, 0.01$.

Solution

Let's derive the relevant expressions by hand, then let MATLAB do the number crunching.

- In general, $\Delta y = f(x+dx) - f(x)$. So for $x = 1$, we have $\Delta y = f(1+dx) - f(1) = (1+dx)^4 + (1+dx)^2 - 2$.
- For $x = 1$, we have $dy = f'(x)\,dx = \left(4x^3 + 2x\right) dx = 6dx$.
- Finally, $\Delta y - dy = (1+dx)^4 + (1+dx)^2 - 6dx - 2$.

```
%---------------------------------------------------
% Stewart 227/18
%
Delta_y = []; Dy = []; Err = []; Dx = [1 0.5 0.1 0.01];
for dx = Dx
    %
    delta_y = (1+dx)^4 + (1+dx)^2 - 2;
    dy = 6*dx;
    err = delta_y - dy;
    %
    Delta_y = [Delta_y delta_y];
    Dy = [Dy dy];
    Err = [Err err];
end
table = [Dx' Delta_y' Dy' Err']
disp('      dx      delta_y       dy     difference')
%
echo off; diary off
%--------------------
table =
    1.0000   18.0000    6.0000   12.0000
    0.5000    5.3125    3.0000    2.3125
    0.1000    0.6741    0.6000    0.0741
    0.0100    0.0607    0.0600    0.0007
      dx     delta_y      dy     difference
```

s227x52

Let $f(x) = \dfrac{1}{1+x^2} = \left(1+x^2\right)^{-1}$.

(a) Find the linear and quadratic approximations to $f(x)$ near $x = 1$.

(b) Illustrate part (a) by graphing f and both approximations on the same figure.

(c) Determine the values of x for which the linear approximation is accurate to within 0.1.

(d) Determine values of x for which the quadratic approximation is accurate to within 0.1.

(e) Compare the exact values of $f(x)$ for $x = 0.9, 1.1, 1.2, 1.3$ with the linear and quadratic approximations for these values of x.

Solution

- Compute the first two derivatives of f.

$$\begin{aligned} f'(x) &= -\left(1+x^2\right)^{-2}(2x) \\ f''(x) &= -\left(-2(1+x^2)^{-3}(2x)(2x) + (1+x^2)^{-2}(2)\right) \end{aligned}$$

- Evaluate these derivatives at $x = 1$.

$$\begin{aligned} f'(1) &= -\frac{1}{2} \\ f''(1) &= \frac{1}{2} \end{aligned}$$

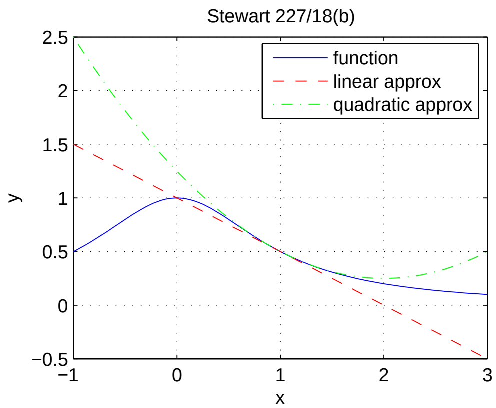
(a) The linear approximation is

$$\begin{aligned} L(x) &= f(1) + f'(1)(x-1) \\ &= \tfrac{1}{2} - \tfrac{1}{2}(x-1). \end{aligned}$$

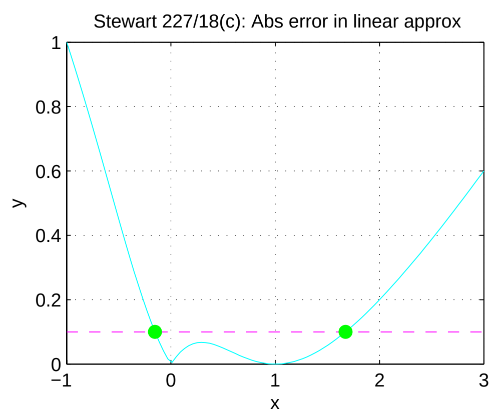
The quadratic approximation is

$$\begin{aligned} Q(x) &= f(1) + f'(1)(x-1) + \tfrac{1}{2} f''(1)(x-1)^2 \\ &= \tfrac{1}{2} - \tfrac{1}{2}(x-1) + \tfrac{1}{4}(x-1)^2. \end{aligned}$$

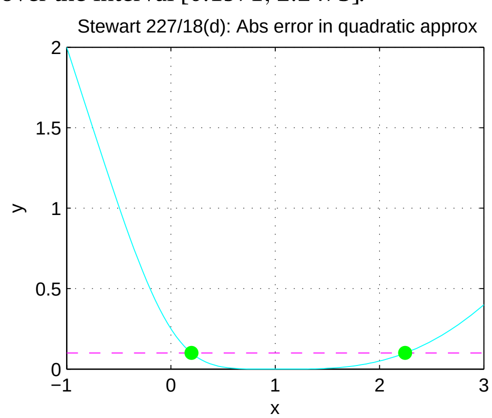
(b) Here are graphs of $f(x)$ and the two approximations.



(c) The linear approximation $L(x)$ is accurate to within 0.1 over the interval $[-0.1538, 1.6740]$.



(d) The quadratic approximation $Q(x)$ is accurate to within 0.1 over the interval $[0.1971, 2.2473]$.



(e) Here are values of $f(x)$, $L(x)$, and $Q(x)$ for four values of x.

x	$f(x)$	$L(x)$	$Q(x)$
0.9	0.5525	0.5500	0.5525
1.1	0.4525	0.4500	0.4525
1.2	0.4098	0.4000	0.4100
1.3	0.3717	0.3500	0.3725

Here is a MATLAB diary file showing all computations and graphics commands.

```
%--------------------------------------------------
% Stewart 228/52
%
% (b)
x = linspace(-1, 3);
tenth = 0.1 + 0*x;
yf = f(x);
yL = L(x);
yQ = Q(x);
plot(x,yf, x,yL,'r--', x,yQ,'g-.'); grid on
legend('function', 'linear approx', 'quadratic approx')
xlabel('x'); ylabel('y'); title('Stewart 227/18(b)')
% (c)
errL = abs(f(x) - L(x));
La = fzero(@errL_cut, [-0.5 0])
La =
   -0.1538
Lb = fzero(@errL_cut, [1.5 2])
Lb =
    1.6740
figure
plot(x,errL,'c', x,tenth,'m--'); grid on; hold on
plot(La, 0.1, 'go', 'MarkerFaceColor', 'g', ...
    'MarkerSize', 7)
plot(Lb, 0.1, 'go', 'MarkerFaceColor', 'g', ...
    'MarkerSize', 7)
xlabel('x'); ylabel('y');
title('Stewart 227/18(c): Abs error in linear approx')
% (d)
errQ = abs(f(x) - Q(x));
Qa = fzero(@errQ_cut, [0 0.5])
Qa =
    0.1971
Qb = fzero(@errQ_cut, [1.5 2.5])
Qb =
    2.2473
figure
plot(x,errQ,'c', x,tenth,'m--'); grid on; hold on
plot(Qa, 0.1, 'go', 'MarkerFaceColor', 'g', ...
    'MarkerSize', 7)
plot(Qb, 0.1, 'go', 'MarkerFaceColor', 'g', ...
    'MarkerSize', 7)
xlabel('x'); ylabel('y');
title('Stewart 227/18(d): Abs error in quadratic approx')
% (e)
for x = [0.9 1.1 1.2 1.3]
    [f(x) L(x) Q(x)]
end % [Output hand-edited afterward for brevity.]
    0.5525    0.5500    0.5525
    0.4525    0.4500    0.4525
    0.4098    0.4000    0.4100
    0.3717    0.3500    0.3725
echo off; diary off
%
%--------------------
function y = f(x)
y = 1 ./ (1 + x.^2);
%--------------------
function y = L(x)
y = 1./2  -  (x-1) ./ 2;
%--------------------
function y = Q(x)
y = 1./2  -  (x-1) ./ 2  +  (x-1).^2 ./ 4;
%--------------------
function y = errL_cut(x)
y = abs(f(x) - L(x)) - 0.1;
%--------------------
function y = errQ_cut(x)
y = abs(f(x) - Q(x)) - 0.1;
```