Fall 2004 Math 151
4 Inverse Functions
4.1 Exponential Functions and Their Derivatives
Mon, 18/Oct ©2004, Art Belmonte

Summary

- **Exponential function**: A function of the form a^x where a is a positive constant and x varies over all real numbers. The number a is called the **base** of the exponential function.
- **Laws of exponents**

  $$a^x a^y = a^{x+y} \qquad \frac{a^x}{a^y} = a^{x-y} \qquad \left(a^x\right)^y = a^{xy} \qquad (ab)^x = a^x b^x$$

- **Continuity / domain / range**: An exponential function with base $a \neq 1$ is continuous with domain $\mathbb{R}$ and range $(0, \infty)$. For $a > 1$ (the typical case), a^x is strictly increasing, whereas for $0 < a < 1$, a^x is strictly decreasing.
- **Limits at infinity**
  - For $a > 1$, $\lim\limits_{x\to\infty} a^x = \infty$ and $\lim\limits_{x\to-\infty} a^x = 0$.
  - For $0 < a < 1$, $\lim\limits_{x\to\infty} a^x = 0$ and $\lim\limits_{x\to-\infty} a^x = \infty$.
- **A special base** is the positive constant e (approximately equal to 2.71828) such that $\lim\limits_{h\to 0} \dfrac{e^h - 1}{h} = 1$.
- **Derivatives**: Let $u = u(x)$ be a differentiable function of x. Then

  $$\frac{d}{dx}\left(e^x\right) = e^x \qquad \frac{d}{dx}\left(e^u\right) = e^u \frac{du}{dx}$$

Hand Examples

249/16

Find $\lim\limits_{x\to\infty} \left(\dfrac{2\pi}{7}\right)^x$.

Solution

Recall that π is approximately equal to 3.14. Thus $a = 2\pi/7 < 1$, whence $\lim\limits_{x\to\infty} \left(\dfrac{2\pi}{7}\right)^x = 0$.

249/22

Find $\lim\limits_{x\to 0^-} \dfrac{2}{1 + e^{\cot x}}$.

Solution

As $x \to 0^-$, we have $\cot x = \dfrac{\cos x}{\sin x} \to -\infty$ (since $\sin x \to 0$ through negative values and $\cos x \to 1$). Therefore,

$$\frac{2}{1 + e^{\cot x}} \to \frac{2}{1 + 0} = 2.$$

249/27

Differentiate $f(x) = e^{\sqrt{x}}$.

Solution

Now $f(x) = e^{x^{1/2}}$. Thus $f'(x) = e^{x^{1/2}} \left(\dfrac{1}{2} x^{-1/2}\right) = \dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$.

249/30

Differentiate $g(x) = e^{-5x} \cos 3x$.

Solution

Use the product and chain rules.

$$\begin{aligned} \frac{dg}{dx} &= \left(-5e^{-5x}\right) \cos 3x + e^{-5x} \left(-3 \sin 3x\right) \\ &= -e^{-5x} \left(5 \cos 3x + 3 \sin 3x\right) \end{aligned}$$

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Differentiate $y = \sec\left(e^{\tan\left(x^2\right)}\right)$.

Solution

Recursive application of the chain rule yields

$$\begin{aligned} y' &= \sec\left(e^{\tan\left(x^2\right)}\right) \tan\left(e^{\tan\left(x^2\right)}\right) \cdot \left(e^{\tan\left(x^2\right)} \sec^2\left(x^2\right) \cdot (2x)\right) \\ &= 2x e^{\tan\left(x^2\right)} \sec^2\left(x^2\right) \sec\left(e^{\tan\left(x^2\right)}\right) \tan\left(e^{\tan\left(x^2\right)}\right) \end{aligned}$$

249/45

Find y' if $\cos(x - y) = x e^x$.

Solution

Use implicit differentiation.

$$\begin{aligned} -\sin(x-y)\cdot\left(1-y'\right) &= e^x + xe^x \\ y'\sin(x-y) &= e^x(x+1) + \sin(x-y) \\ y' &= e^x(x+1)\csc(x-y) + 1 \end{aligned}$$

250/57

Evaluate $\lim\limits_{x\to\pi} \dfrac{e^{\sin x} - 1}{x - \pi}$.

Solution

We recognize $\lim\limits_{x\to\pi} \dfrac{e^{\sin x} - e^{\sin \pi}}{x - \pi}$ as the definition of the derivative of $f(x) = e^{\sin x}$ at $x = \pi$. Hence

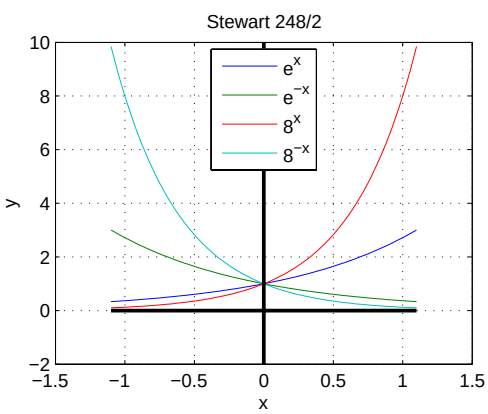
$$\begin{aligned} f'(x) &= e^{\sin x}\cos x \\ f'(\pi) &= e^{\sin \pi}\cos \pi = e^0(-1) = -1. \end{aligned}$$

MATLAB Examples

s248x02

Graph the exponential functions e^x, e^{-x}, 8^x, 8^{-x} on the same figure. Note that each passes through the point (0, 1).

Solution

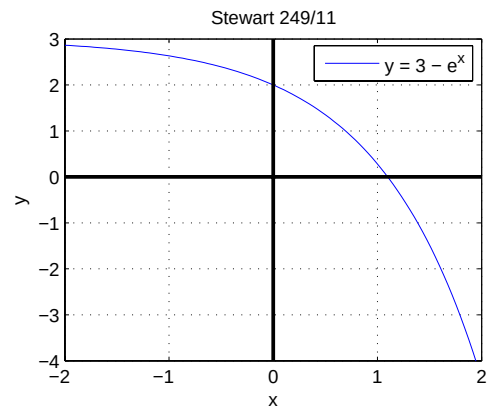



```
%-----------------------------------------------
% Stewart 248/2
%
x = linspace(-1.1, 1.1);
y1 = exp(x); y2 = exp(-x); y3 = 8.^x; y4 = 8.^(-x);
plot(x,y1, x,y2, x,y3, x,y4); grid on; hold on
legend('e^x', 'e^{-x}', '8^x', '8^{-x}', 'Location', 'North')
plot([-1.1 1.1], [0 0], 'k', 'LineWidth', 2)
plot([0 0], [-2 10], 'k', 'LineWidth', 2)
xlabel('x'); ylabel('y')
title('Stewart 248/2')
%
echo off; diary off
```

s249x11

Plot $y = 3 - e^x$.

Solution



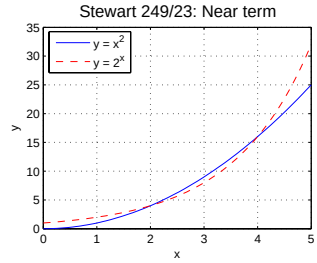

```
%-----------------------------------------------
% Stewart 249/11
%
x = linspace(-2, 2);
y = 3 - exp(x);
plot(x,y); grid on; hold on
legend('y = 3 - e^x', 'Location', 'NorthEast')
plot([-2 2], [0 0], 'k', 'LineWidth', 2)
plot([0 0], [-4 3], 'k', 'LineWidth', 2)
xlabel('x'); ylabel('y')
title('Stewart 249/11')
axis([-2 2 -4 3])
%
echo off; diary off
```

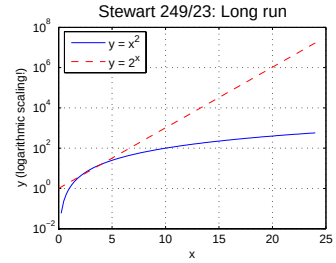
s249x23

Show that if the graphs of $f(x) = x^2$ and $g(x) = 2^x$ are drawn on a coordinate grid where the unit of measurement is 1 inch, then at a distance of 2 feet (or 24 inches) to the right of the origin the height of the graph of f is 48 feet, but the height of the graph of g is about 265 miles.

Solution

In the near term, the polynomial function x^2 and the exponential function 2^x have values of similar magnitude. However, in the long run any exponential function will far outstrip any polynomial function; hence the phrase "exponential growth!"





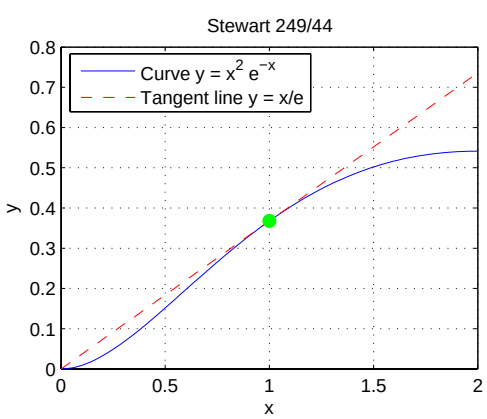
```
%-------------------------------------------------
% Stewart 249/23
%
% NEAR TERM: Polynomial and exponential functions
% have values of similar magnitude.
%
x = linspace(0, 5);
y1 = x.^2; y2 = 2.^x;
plot(x,y1, x,y2,'r--'); grid on
legend('y = x^2', 'y = 2^x', 'Location', 'NorthWest')
xlabel('x'); ylabel('y')
title('Stewart 249/23: Near term')
%
% LONG RUN: The expoential function grows massively
% faster than the polynomial function; hence the phrase
% "exponential growth."
%
figure
x = linspace(0, 24);
y1 = x.^2; y2 = 2.^x;
semilogy(x,y1, x,y2,'r--'); grid on
legend('y = x^2', 'y = 2^x', 'Location', 'NorthWest')
xlabel('x'); ylabel('y')
title('Stewart 249/23: Long run')
%
y1_24 = 24^2, y2_24 = 2^24 % y-values in inches
y1_24 =
   576
y2_24 =
    16777216
y2_24_in_miles = y2_24 / 12 / 5280
y2_24_in_miles =
  264.7919
%
echo off; diary off
```

s249x44

Find an equation of the tangent line to the curve $y = x^2 e^{-x}$ at $(1, 1/e)$.

Solution

Now $y' = 2xe^{-x} - x^2 e^{-x} = 1/e$ at $x = 1$. So the tangent line is $y - \frac{1}{e} = \frac{1}{e}(x-1)$ or $y = \frac{1}{e}x$. Here is a plot of the curve and tangent line.



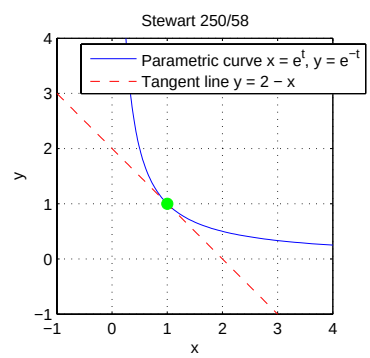

```
%-------------------------------------------------
% Stewart 249/44
%
x = linspace(0, 2);
y = x.^2 .* exp(-x);
yL = x ./ exp(1);
plot(x,y, x,yL,'r--'); grid on; hold on
plot(1, 1/exp(1), 'go', 'MarkerFaceColor', 'g', ...
    'Markersize', 7)
legend('Curve y = x^2 e^{-x}', ...
    'Tangent line y = x/e', 'Location', 'NorthWest')
xlabel('x'); ylabel('y')
title('Stewart 249/44')
%
echo off; diary off
```

s250x58

Plot the parametric curve $x = e^t$, $y = e^{-t}$ for $-1 \le t \le 1$. Find an equation of the tangent line to this curve at the point corresponding to $t = 0$.

Solution

- A point on the tangent line is $(x(0), y(0)) = (1, 1)$.
- Compute parametric derivatives: $\dfrac{dx}{dt} = e^t$ and $\dfrac{dy}{dt} = -e^{-t}$.
- At $t = 0$, the slope of the tangent line is
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-e^0}{e^0} = -1.$$
- Therefore, an equation of the tangent line is
$$\begin{aligned} y - 1 &= -1(x-1) \\ y &= 2 - x. \end{aligned}$$




```
%-------------------------------------------------
% Stewart 250/58
%
t = linspace(-1.5, 1.5);
x = exp(t); y = exp(-t);
plot(x,y); grid on; hold on;
axis equal; axis([-1 4 -1 4])
%
xL = linspace(-1, 3);
yL = 2 - xL;
plot(xL,yL, 'r--')
plot(1, 1, 'go', 'MarkerFaceColor', 'g', ...
    'Markersize', 7)
legend('Parametric curve x = e^t, y = e^{-t}', ...
    'Tangent line y = 2 - x', 'Location', 'NorthEast')
xlabel('x'); ylabel('y')
title('Stewart 250/58')
%
echo off; diary off
```