Fall 2004 Math 151
4 Inverse Functions
4.2 Inverse Functions
Wed, 20/Oct ©2004, Art Belmonte

Summary

- **One-to-one (1-1) function**: A function $f(x)$ for which no two elements of the domain of f have the same image; i.e., $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. Equivalently, a function $f(x)$ such that $f(x_1) = f(x_2)$ implies $x_1 = x_2$. (Each way of saying it means that no two elements of the domain of f have the same image.) Graphically, no horizontal line intersects the graph of f more that once (the so-called **Horizontal Line Test**).
- **Inverse function**: For a 1-1 function f with domain A and range B, its inverse function f^{-1} (with swapped domain B and range A) is given by

$$f^{-1}(y) = x \text{ if and only if } f(x) = y \text{ for } y \in B.$$

- **Cancellation equations**

$$\begin{aligned} f^{-1}\Big(f(x)\Big) &= x, \quad x \in A \\ f\Big(f^{-1}(y)\Big) &= y, \quad y \in B \end{aligned}$$

- **Analytically finding the inverse of a 1-1 function**
  (a) Solve $y = f(x)$ for x.
  (b) Swap x and y in the resulting solution.
  (c) This results in $y = f^{-1}(x)$.
- **Geometrically finding the inverse of a 1-1 function**
  (a) Draw the 45° line, $y = x$.
  (b) The graph of f^{-1} is obtained by reflecting the graph of f across the line $y = x$.
- **Parametrically finding the inverse of a 1-1 function**
  (a) Given $y = f(x)$, let $x = t$ and $y = f(t)$. Then $f = \{(t, f(t)) : t \in A\}$, where A is the domain of f.
  (b) The inverse is given by $f^{-1} = \{(f(t), t) : t \in A\}$. (Basically, swapping x and y amounts to reflecting the graph of f across the 45°line.)
- **THEOREM**: A 1-1 continuous function on an interval has a continuous inverse function.
- **THEOREM**: Let f be a 1-1 differentiable function with inverse $g = f^{-1}$. Then $g'(a) = \dfrac{1}{f'(g(a))}$, provided that the denominator is nonzero.

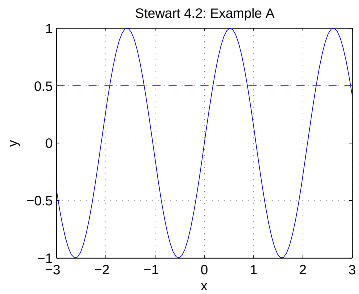
Hand Examples

Example A

Is the function $y = \sin 3x$ a 1-1 function?

Solution

No: it fails the Horizontal Line Test.

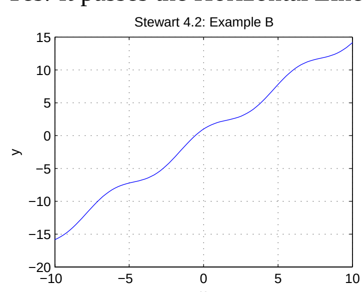


Example B

Is the function $y = 1.5x + \cos x$ a 1-1 function?

Solution

Yes: it passes the Horizontal Line Test.



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Is the function $g(x) = \sqrt{x}$ a 1-1 function?

Solution

Suppose $g(x_1) = g(x_2)$. Then $\sqrt{x_1} = \sqrt{x_2}$. Hence $x_1 = x_2$ by squaring each side. Therefore, g is 1-1.

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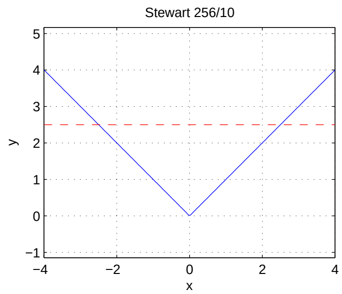
Is the function $y = |x|$ a 1-1 function?

Solution

No: it fails the Horizontal Line Test.



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Show that $f(x) = 5 - 4x^3$ is 1-1 and find its inverse function.

Solution

Now $f(x_1) = f(x_2)$ implies $5 - 4x_1^3 = 5 - 4x_2^3$, whence $x_1 = x_2$ (sticking to real numbers). Therefore f is 1-1.

- Solve $y = f(x) = 5 - 4x^3$ for x to obtain $x = \sqrt[3]{\frac{1}{4}(5 - y)}$.
- Swap x and y in the preceding result: $y = \sqrt[3]{\frac{1}{4}(5 - x)}$.
- Therefore $f^{-1}(x) = \sqrt[3]{\frac{1}{4}(5 - x)}$.

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Find $g'(4)$, where $g = f^{-1}$ is the inverse function of $f(x) = 3 + x + e^x$.

Solution

Unlike the preceding problem, we cannot explicitly obtain an expression for $g(x) = f^{-1}(x)$. Nevertheless, we may still compute the requested derivative. (Note that $f'(x) = 1 + e^x > 0$, so that f is always increasing. Accordingly, its graph passes the Horizontal Line Test and thus f is 1-1.)

By inspection, we have $f(0) = 4$, whence $g(4) = f^{-1}(4) = 0$. Therefore, by the last theorem in the Summary we have

$$g'(4) = \frac{1}{f'(g(4))} = \frac{1}{f'(0)} = \frac{1}{2}.$$

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Suppose $g = f^{-1}$ is the inverse function of a differentiable function f and let $G(x) = 1/g(x)$. If $f(3) = 2$ and $f'(3) = \frac{1}{9}$, find $G'(2)$.

Solution

Now $G'(x) = -(g(x))^{-2}\, g'(x)$. Next, since $f(3) = 2$, we have $g(2) = f^{-1}(2) = 3$. Therefore,

$$G'(2) = -\frac{g'(2)}{(g(2))^2} = -\frac{\frac{1}{f'(g(2))}}{(g(2))^2} = -\frac{\frac{1}{f'(3)}}{(3)^2} = -\frac{\frac{1}{1/9}}{9} = -1.$$

MATLAB Examples

s256x21

Let $f(x) = x^3$ and $a = 8$.

(a) Show that f is 1-1.

(b) Find $g'(a)$ where $g = f^{-1}$.

(c) Calculate $g(x)$ explicitly and state its domain and range.

(d) Recalculate $g'(a)$ from the formula for $g(x)$ in part (c) and check that it agrees with the result in part (b).

(e) Sketch the graphs of f and $g = f^{-1}$ on the same figure.

Solution

(a) Now $f(x_1) = f(x_2)$ implies $x_1^3 = x_2^3$, whence $x_1 = x_2$. Thus f is 1-1.

(b) By inspection $f(2) = 8$. So $g(8) = f^{-1}(8) = 2$. Note that $f'(x) = 3x^2$. By the last theorem in the Summary, we have

$$g'(a) = g'(8) = \frac{1}{f'(g(8))} = \frac{1}{f'(2)} = \frac{1}{3\,(2)^2} = \frac{1}{12}.$$

(c) Determine $g(x) = f^{-1}(x)$.

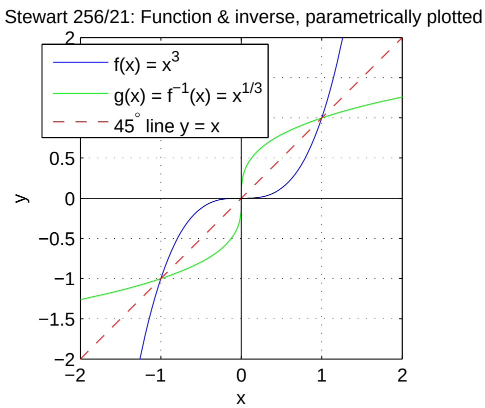
- Solve $y = f(x) = x^3$ for x to obtain $x = \sqrt[3]{y}$.
- Swap x and y in the preceding result: $y = \sqrt[3]{x}$.
- Therefore $g(x) = f^{-1}(x) = \sqrt[3]{x} = x^{1/3}$.

(d) Hence $g'(x) = \frac{1}{3}x^{-2/3}$. Thus

$$g'(a) = g'(8) = \frac{1}{3}\,(8)^{-2/3} = \frac{1}{3}\left(\frac{1}{4}\right) = \frac{1}{12},$$

as before.

(e) Graphs of f and $g = f^{-1}$ appear together in the following plot, along with the 45° line $y = x$. Observe how f and f^{-1} are mirror reflections of one another across the 45° line.




```
%-------------------------------------------------
% Stewart 256/21
%
t = linspace(-2, 2);
x = t; y = t.^3;
%
plot(x,y, y,x,'g', t,t,'r--'); grid on; hold on
legend('f(x) = x^3', 'g(x) = f^{-1}(x) = x^{1/3}', ...
    '45^{\circ} line y = x', 'Location', 'NorthWest')
xlabel('x'); ylabel('y')
title(['Stewart 256/21: Function & inverse, ' ...
    'parametrically plotted'])
axis equal; axis([-2 2 -2 2])
% Axes
plot([-2 2], [0 0], 'k')
plot([0 0], [-2 2], 'k')
%
echo off; diary off
```

s257x37

The equation $\sqrt[5]{x} - \sqrt[5]{y} = y$ defines y implicitly as a function of x: $y = f(x)$.

(a) Find an explicit expression for $g(x) = f^{-1}(x)$, the inverse of $f(x)$.

(b) Graph f using a parametric graph.

Solution

(a) Since $y = f(x)$ [implicitly] defines y in terms of x, then $x = f^{-1}(y)$ defines x in terms of y. We have

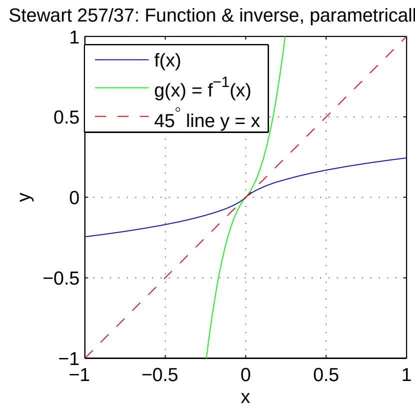
$$x = \left(y + \sqrt[5]{y}\right)^5 = f^{-1}(y) = g(y)$$

Thus $g(x) = f^{-1}(x) = \left(x + \sqrt[5]{x}\right)^5$.

(b) Graph f by graphing $(x(t), y(t)) = \left(\left(t + \sqrt[5]{t}\right)^5, t\right)$. The graph is symmetric with respect to the origin. To see this, let's show that g is an odd function. Let $a > 0$. Then,

$$\begin{aligned} g(-a) &= \left(-a + \sqrt[5]{-a}\right)^5 \\ &= \left(-a - \sqrt[5]{a}\right)^5 \\ &= -\left(a + \sqrt[5]{a}\right)^5 \\ &= -g(a). \end{aligned}$$

A slick way of coding an odd function $h(t)$ in MATLAB is **sign(t) .* h(abs(t))**.




```
%-------------------------------------------------
% Stewart 257/37
%
t = linspace(-1, 1);
x = g(t);  y = t;
%
plot(x,y, y,x,'g', t,t,'r--'); grid on; hold on
legend('f(x)', 'g(x) = f^{-1}(x)', ...
    '45^{\circ} line y = x', 'Location', 'NorthWest')
xlabel('x'); ylabel('y')
title(['Stewart 257/37: Function & inverse, ' ...
    'parametrically plotted'])
axis equal; axis([-1 1 -1 1])
%
echo off; diary off
%--------------------
function z = g(t)
a = abs(t);
z = sign(t) .* abs(a + a.^(1/5)) .^ 5;
```