Fall 2004 Math 151
4 Inverse Functions
4.6 Inverse Trigonometric Functions
Fri, 29/Oct ©2004, Art Belmonte

Summary

Definitions

With suitable domain restrictions, one may define inverses of trigonometric functions. The symbol $\Longleftrightarrow$ stands for an implication that goes both ways ("if and only if").

1. **Inverse sine (arcsine)**: $\sin^{-1}$ or arcsin
$$y = \sin^{-1} x \Longleftrightarrow \left(\sin y = x \text{ and } |y| \le \frac{\pi}{2}\right)$$
2. **Inverse cosine (arccosine)**: $\cos^{-1}$ or arccos
$$y = \cos^{-1} x \Longleftrightarrow (\cos y = x \text{ and } 0 \le y \le \pi)$$
3. **Inverse tangent (arctangent)**: $\tan^{-1}$ or arctan
$$y = \tan^{-1} x \Longleftrightarrow \left(\tan y = x \text{ and } |y| < \frac{\pi}{2}\right)$$
4. **Inverse cotangent (arccotangent)**: $\cot^{-1}$ or arccot
$$y = \cot^{-1} x \ (x \in \mathbb{R}) \Longleftrightarrow (\cot y = x \text{ and } y \in (0, \pi))$$
5. **Inverse secant (arcsecant)**: $\sec^{-1}$ or arcsec
$$y = \sec^{-1} x \ (|x| \ge 1) \Longleftrightarrow \left(\begin{array}{l} \sec y = x \text{ and} \\ y \in [0, \frac{\pi}{2}) \cup [\pi, \frac{3}{2}\pi) \end{array}\right)$$
6. **Inverse cosecant (arccosecant)**: $\csc^{-1}$ or arccsc
$$y = \csc^{-1} x \ (|x| \ge 1) \Longleftrightarrow \left(\begin{array}{l} \csc y = x \text{ and} \\ y \in (0, \frac{\pi}{2}] \cup (\pi, \frac{3}{2}\pi] \end{array}\right)$$

Some cancellation equations

- $\sin^{-1}(\sin x) = x$ for $|x| \le \frac{\pi}{2}$
- $\sin\left(\sin^{-1} x\right) = x$ for $|x| \le 1$
- $\cos^{-1}(\cos x) = x$ for $0 \le x \le \pi$
- $\cos\left(\cos^{-1} x\right) = x$ for $|x| \le 1$
- $\tan^{-1}(\tan x) = x$ for $|x| < \frac{\pi}{2}$
- $\tan\left(\tan^{-1} x\right) = x$ for $x \in \mathbb{R}$

Some limits

- $\displaystyle\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$
- $\displaystyle\lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}$

Six derivatives

These may be combined with the chain rule.

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1+x^2}$$
$$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}\left(\csc^{-1} x\right) = -\frac{1}{x\sqrt{x^2-1}}$$

MATLAB commands

Math	$\sin^{-1}$	$\cos^{-1}$	$\tan^{-1}$	$\cot^{-1}$	$\sec^{-1}$	$\csc^{-1}$
MATLAB	**asin**	**acos**	**atan**	**acot**	**asec**	**acsc**

Hand Examples

281/1

Find the exact value of $\cos^{-1}(-1)$.

Solution

Now $y = \cos^{-1}(-1) \Longleftrightarrow (\cos y = -1 \text{ and } 0 \le y \le \pi)$. Hence $y = \pi$. ("The angle between 0 and π whose cosine is -1 is π.")

281/2

Find the exact value of $\sin^{-1}\left(\frac{1}{2}\right)$.

Solution

Now $y = \sin^{-1}\left(\frac{1}{2}\right) \Longleftrightarrow \left(\sin y = \frac{1}{2} \text{ and } |y| \le \frac{\pi}{2}\right)$. Hence $y = \frac{\pi}{6}$. ("The angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is $\frac{1}{2}$ is $\frac{\pi}{6}$.")

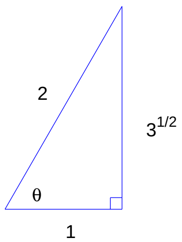
281/12

Find the exact value of $\tan\left(\cos^{-1}\left(\frac{1}{2}\right)\right)$.

Solution

"What is the tangent of the angle whose cosine is $\frac{1}{2}$?"

Draw a right triangle! We have $\tan\theta = \dfrac{\text{opp}}{\text{adj}} = \dfrac{\sqrt{3}}{1} = \sqrt{3}$.



281/15

Find the exact value of $\arcsin\left(\sin\left(\frac{5}{4}\pi\right)\right)$.

Solution

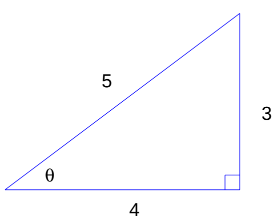
We have $\sin^{-1}\left(\sin\left(\frac{5}{4}\pi\right)\right) = \sin^{-1}\left(-\frac{1}{2}\sqrt{2}\right) = -\frac{\pi}{4}$ by the first cancellation equation.

281/16

Find the exact value of $\sin\left(2\sin^{-1}\frac{3}{5}\right)$.

Solution

Let $\theta = \sin^{-1}\frac{3}{5}$. By the first cancellation equation, $0 \le \theta \le \frac{\pi}{2}$. Draw a triangle. Then $\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right) = \frac{24}{25}$.

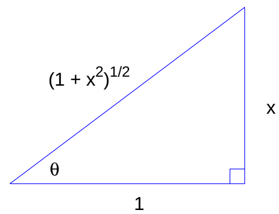


281/21

Simplify the expresson $\sin\left(\tan^{-1} x\right)$.

Solution

Draw a right triangle. Then $\sin\left(\tan^{-1} x\right) = \dfrac{x}{\sqrt{1+x^2}}$.



282/28

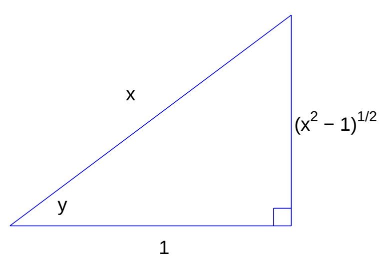
Derive the formula for the derivative of $\sec^{-1} x$.

Solution

For $|x| \ge 1$, let $y = \sec^{-1} x$. Then $\sec y = x$. Hence

$$\begin{aligned} \sec y \tan y \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\sec y \tan y} \\ \frac{d}{dx}\left(\sec^{-1} x\right) &= \frac{1}{x\sqrt{x^2-1}}, \quad |x| > 1. \end{aligned}$$

The last step is readily seen by drawing a triangle.



282/32

Find the derivative of $y = \left(\sin^{-1} x\right)^2$.

Solution

Via the chain rule, we have

$$y' = 2\left(\sin^{-1} x\right)\frac{1}{\sqrt{1-x^2}} = \frac{2\sin^{-1} x}{\sqrt{1-x^2}}.$$

282/33

Find the derivative of $y = \sin^{-1}\left(x^2\right)$.

Solution

We have $y' = \dfrac{1}{\sqrt{1-(x^2)^2}}(2x) = \dfrac{2x}{\sqrt{1-x^4}}$.

282/43

Let $y = \tan^{-1}(\sin x)$. Compute dy/dx.

Solution

We have $\dfrac{dy}{dx} = \dfrac{1}{1+(\sin x)^2} \cdot \cos x = \dfrac{\cos x}{1+\sin^2 x}$.

282/47

If $y = x^2 \cot^{-1}(3x)$, find y'.

Solution

We have

$$\begin{aligned} y' &= 2x\cot^{-1}(3x) + x^2\left(-\frac{1}{1+(3x)^2}(3)\right) \\ &= 2x\cot^{-1}(3x) - \frac{3x^2}{1+9x^2}. \end{aligned}$$

282/52

Let $f(x) = \sqrt{\sin^{-1}(2/x)}$. Find $f'(x)$ and state the domains of f and f'.

Solution

- Rewrite f as $f(x) = \left(\sin^{-1}\left(2x^{-1}\right)\right)^{1/2}$. Then

$$\begin{aligned} f'(x) &= \frac{1}{2}\left(\sin^{-1}\left(2x^{-1}\right)\right)^{-1/2} \frac{1}{\sqrt{1-\left(2x^{-1}\right)^2}}\left(-2x^{-2}\right) \\ &= -\frac{1}{x^2\sqrt{\sin^{-1}\left(\frac{2}{x}\right)}\sqrt{1-\frac{4}{x^2}}}. \end{aligned}$$

- For f, we require the argument of the square root to be nonnegative. Now $0 \le \sin^{-1}(2/x) \le \frac{\pi}{2}$ in turn implies that $0 \le 2/x \le 1$, whence $2 \le x$. The domain of f is therefore $[2,\infty)$.

- The domain of f' is similar except that we must exclude $x = 2$ lest we divide by zero. Therefore, the domain of $f'$is $(2,\infty)$.

282/62

Compute the limit $\lim_{x\to\infty} \sin^{-1}\left(\dfrac{x+1}{2x+1}\right)$.

Solution

We have $\lim_{x\to\infty} \sin^{-1}\left(\dfrac{x+1}{2x+1}\right) = \lim_{x\to\infty} \sin^{-1}\left(\dfrac{1+\frac{1}{x}}{2+\frac{1}{x}}\right) = \dfrac{\pi}{6}$.

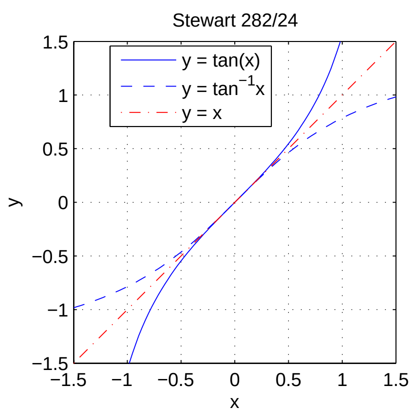
MATLAB Examples

s282x24

Graph $y = \tan x$, $y = \tan^{-1} x$, and $y = x$ on the same figure. How are the first two of these related?

Solution

The two functions $\tan x$ and $\tan^{-1}$x are inverse functions. Hence their graphs are mirror reflections of one another across the 45° line $y = x$.




```
%-------------------------------------------------
% Stewart 282/24
%
x = linspace(-1.5, 1.5);
y1 = tan(x); y2 = atan(x); y3 = x;
plot(x,y1,'b', x,y2,'b--', x,y3,'r-.')
grid on; axis equal; axis([-1.5 1.5 -1.5 1.5])
legend('y = tan(x)', 'y = tan^{-1}x', 'y = x', ...
    'Location', 'North')
xlabel('x'); ylabel('y'); title('Stewart 282/24')
%
echo off; diary off
```

s282x41

Find the derivative of $y = \sec^{-1}\sqrt{1+x^2}$.

Solution

MATLAB's **diff** renders the needful. The simplified answer is equivalent to $\dfrac{x}{|x|\left(1+x^2\right)}$.

```
%-------------------------------------------------
% Stewart 282/41
%
syms x
y = asec(sqrt(1 + x^2)); pretty(y)

                                        2 1/2
                             asec((1 + x )    )
dy_dx = diff(y,x); pretty(dy_dx)

                                          x
                          --------------------------
                                2 3/2 /       1   \1/2
                          (1 + x )    |1 - ------|
                                      |        2|
                                      \    1 + x /
dy_dx = simplify(dy_dx); pretty(dy_dx)

                                          x
                            ----------------------
                                          /   2  \1/2
                                  2 3/2 |  x    |
                            (1 + x )    |------|
                                        |      2|
                                        \1 + x /
%
echo off; diary off
```

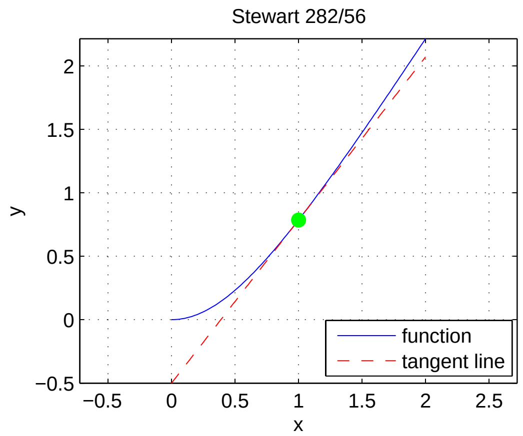
s282x56

Let $f(x) = x\tan^{-1}x$. Find $f'(1)$. [Additionally, find an equation of the tangent line to the graph of f at $\left(1, \frac{\pi}{4}\right)$. Make a plot.]

Solution

We see that $f'(1) = \frac{1}{4}\pi + \frac{1}{2}$. The tangent line is

$$y = \tfrac{1}{4}\pi + \left(\tfrac{1}{4}\pi + \tfrac{1}{2}\right)(x-1).$$



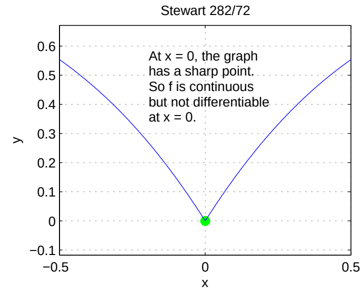

```
%-------------------------------------------------
% Stewart 282/56
%
syms x
f = x * atan(x);
f1 = subs(f, x, sym(1)); pretty(f1)

                                    1/4 pi
fp = diff(f,x); pretty(fp)

                                         x
                              atan(x) + ------
                                             2
                                        1 + x
fp1 = subs(fp, x, sym(1)); pretty(fp1)

                                 1/4 pi + 1/2
TL = f1 + fp1*(x-1); pretty(TL)

                        1/4 pi + (1/4 pi + 1/2) (x - 1)
x = linspace(0, 2);
f = eval(vectorize(f));
TL = eval(vectorize(TL));
plot(x,f, x,TL,'r--'); hold on
plot(1, eval(f1), 'go', 'MarkerFaceColor', 'g', ...
    'MarkerSize', 7)
grid on; axis equal
legend('function', 'tangent line', ...
    'Location', 'SouthEast')
xlabel('x'); ylabel('y'); title('Stewart 282/56')
%
echo off; diary off
```

s283x72

Let $f(x) = x\tan^{-1}(1/x)$ if $x \neq 0$ and $f(0) = 0$. Is f continuous at 0? Is f differentiable at 0?

Solution

- We see that $\lim_{x\to 0} f(x) = f(0)$. So f is continuous at 0.
- Now $\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = -\frac{\pi}{2}$, but $\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \frac{\pi}{2}$. Therefore, $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$ does *not* exist. Hence f is *not* differentiable at 0.




```
%-------------------------------------------------
% Stewart 282/72
%
syms x
f = x * atan(1/x); f0 = 0;
L1 = limit(f, x, 0)
   L1 = 0
DQ = (f - f0) / (x - 0);
L2 = limit(DQ, x, 0, 'left')
   L2 = -1/2*pi
L3 = limit(DQ, x, 0, 'right')
   L3 = 1/2*pi
%

echo off; diary off
```