Fall 2004 Math 151
4 Inverse Functions
4.8 Indeterminate Forms and L'Hospital's Rule
Mon, 01/Nov ©2004, Art Belmonte

Summary

Definitions

- **Indeterminate quotient**: An expression that in the limit appears to have one of these forms:

$$\frac{0}{0} \qquad \frac{\infty}{\infty} \qquad \frac{\infty}{-\infty} \qquad \frac{-\infty}{\infty} \qquad \frac{-\infty}{-\infty}.$$

- **Indeterminate product**: An expression that in the limit appears to have one of these forms: $(0)(\infty) \quad (0)(-\infty)$.

- **Indeterminate difference**: An expression that in the limit appears to have the form $\infty - \infty$.

- **Indeterminate power**: An expression that in the limit appears to have one of these forms:

$$0^0 \qquad \infty^0 \qquad 1^\infty \qquad 1^{-\infty}$$

L'Hospital's Rule

Let f and g be differentiable with $g' \neq 0$ on an open interval I (except perhaps at a). Let $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ be an indeterminate quotient. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided that the latter limit exists (or is ∞ or $-\infty$).

NOTES

- Make sure that the hypotheses of L'Hospital's Rule are satisfied before you attempt to apply the rule. If they are not satisfied, then your reasoning is invalid. This may lead to erroneous answers.

- Instead of x approaching a from both sides (a two-sided limit), the rule also applies to one-sided limits and limits at infinity. In other words, it applies when

$$x \to a^- \qquad x \to a^+ \qquad x \to \infty \qquad x \to -\infty.$$

- **To handle indeterminate products or differences**, use algebra to first transform them into indeterminate quotients.

- **To handle indeterminate powers**, proceed as follows.

1. Label the limiting expression as $y = f(x)^{g(x)}$.
2. Take natural logs of each side: $\ln y = g(x) \ln f(x)$.
3. The preceding right-hand side is now an indeterminate product. Evaluate its limiting behavior.

$$\lim (\ln y) = \lim (g(x) \ln f(x)) = L$$

4. We then have $\lim f(x)^{g(x)} = e^L$. Here's why.

$$\lim f(x)^{g(x)} = \lim y = \lim \left(e^{\ln y}\right) = e^{(\lim \ln y)} = e^L$$

where the penultimate step is justfied due to the continuity of the exponential function e^w.

Hand Examples

295/5

Find the limit $\lim\limits_{x\to 0} \dfrac{e^x - 1}{\sin x}$.

Solution

The limiting expression is an indeterminate quotient $0/0$. Therefore, we may apply L'Hospital's Rule, as noted by $\overset{\text{L'H}}{=}$.

$$\lim_{x\to 0} \frac{e^x - 1}{\sin x} \overset{\text{L'H}}{=} \lim_{x\to 0} \frac{e^x}{\cos x} = 1$$

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Find the limit $\lim\limits_{x\to 3\pi/2} \dfrac{\cos x}{x - (3\pi/2)}$.

Solution

This is another indeterminate quotient $0/0$.

$$\lim_{x\to 3\pi/2} \frac{\cos x}{x - (3\pi/2)} \overset{\text{L'H}}{=} \lim_{x\to 3\pi/2} \frac{-\sin x}{1} = 1$$

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Find the limit $\lim\limits_{x\to a} \dfrac{\sqrt[3]{x} - \sqrt[3]{a}}{x - a}$, where $a \neq 0$.

Solution

The form is $0/0$.

$$\lim_{x\to a} \frac{\sqrt[3]{x} - \sqrt[3]{a}}{x - a} \overset{\text{L'H}}{=} \lim_{x\to a} \frac{\frac{1}{3}x^{-2/3}}{1} = \tfrac{1}{3}a^{-2/3}$$

295/20

Find the limit $\displaystyle\lim_{x\to 0} \frac{\sin^2 x}{\tan\left(x^2\right)}$.

Solution

The form is $0/0$.

$$\begin{aligned}\lim_{x\to 0} \frac{\sin^2 x}{\tan\left(x^2\right)} &\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{2\sin x \cos x}{\sec^2\left(x^2\right)\cdot(2x)} \\ &= \lim_{x\to 0}\left(\frac{\sin 2x}{2x}\cos^2\left(x^2\right)\right) \\ &= 1\end{aligned}$$

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Find the limit $\displaystyle\lim_{x\to\infty} \frac{\ln\left(1+e^x\right)}{5x}$.

Solution

The form is ∞/∞, an indeterminate quotient.

$$\lim_{x\to\infty} \frac{\ln\left(1+e^x\right)}{5x} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{\frac{e^x}{1+e^x}}{5} = \lim_{x\to\infty} \frac{\frac{1}{1+e^{-x}}}{5} = \frac{1}{5}$$

295/35

Find the limit $\displaystyle\lim_{x\to 0} \frac{e^{4x}-1}{\cos x}$.

Solution

Plug & chug: $\displaystyle\lim_{x\to 0} \frac{e^{4x}-1}{\cos x} = 0$. (It's *not* an indeterminate form!)

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Find the limit $\displaystyle\lim_{x\to -\infty} xe^x$.

Solution

This is an indeterminate product $(-\infty)(0)$. Rewrite it as an indeterminate quotient, then apply L'Hospital's Rule.

$$\lim_{x\to -\infty} xe^x = \lim_{x\to -\infty} \frac{x}{e^{-x}} \overset{\text{L'H}}{=} \lim_{x\to -\infty} \frac{1}{-e^{-x}} = 0$$

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Find the limit $\displaystyle\lim_{x\to 0}\left(\frac{1}{x} - \csc x\right)$.

Solution

This is an indeterminate difference $\infty - \infty$. Convert this to an indeterminate quotient, then apply L'Hospital's Rule (twice in this case).

$$\begin{aligned}\lim_{x\to 0}\left(\frac{1}{x} - \csc x\right) &= \lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{\sin x}\right) \\ &= \lim_{x\to 0} \frac{\sin x - x}{x\sin x} \\ &\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{\cos x - 1}{\sin x + x\cos x} \\ &\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{-\sin x}{\cos x + \cos x - x\sin x} \\ &= 0\end{aligned}$$

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Find the limit $\displaystyle\lim_{x\to\infty}\left(\frac{x^3}{x^2-1} - \frac{x^3}{x^2+1}\right)$.

Solution

The form is $\infty - \infty$. Convert to an indeterminate quotient.

$$\begin{aligned}\lim_{x\to\infty}\left(\frac{x^3}{x^2-1} - \frac{x^3}{x^2+1}\right) &= \lim_{x\to\infty}\left(\frac{x^3\left(x^2+1\right) - x^3\left(x^2-1\right)}{\left(x^2-1\right)\left(x^2+1\right)}\right) \\ &= \lim_{x\to\infty} \frac{2x^3}{x^4-1} \\ &= \lim_{x\to\infty} \frac{\frac{2}{x}}{1-\frac{1}{x^4}} \\ &= 0\end{aligned}$$

295/56

Find the limit $\displaystyle\lim_{x\to 0^+} (\sin x)^{\tan x}$.

Solution

This is an indeterminate power 0^0. Use the four-step procedure outlined at the end of the Summary.

1. Let $y = (\sin x)^{\tan x}$.
2. Then $\ln y = (\tan x)\,(\ln(\sin x)) = \dfrac{\ln(\sin x)}{\cot x}$,
   an indeterminate quotient $-\infty/\infty$ as $x \to 0^+$.
3. Therefore,

$$\begin{aligned}
\lim_{x\to 0^+} \frac{\ln(\sin x)}{\cot x} &\overset{\text{L'H}}{=} \lim_{x\to 0^+} \frac{\cos x/\sin x}{-\csc^2 x} \\
&= \lim_{x\to 0^+} (-\cos x \sin x) \\
\lim_{x\to 0^+} \ln y &= 0 \qquad \text{[from \#2 label } y\text{]}.
\end{aligned}$$

4. Hence $\lim\limits_{x\to 0^+} (\sin x)^{\tan x} = \lim\limits_{x\to 0^+} y = \lim\limits_{x\to 0^+} e^{\ln y} = e^0 = 1$.

295/58

Find the limit $\lim\limits_{x\to\infty} \left(1 + \dfrac{a}{x}\right)^{bx}$, where a and b are constants.

Solution

This is an indeterminate power 1^∞ or $1^{-\infty}$. Use the four-step procedure outlined at the end of the Summary.

1. Let $y = \left(1 + \dfrac{a}{x}\right)^{bx}$.
2. Then $\ln y = (bx)\left(\ln\left(1 + \dfrac{a}{x}\right)\right) = \dfrac{b\ln\left(1 + ax^{-1}\right)}{x^{-1}}$,
   an indeterminate quotient $0/0$ as $x \to \infty$.
3. Therefore,

$$\begin{aligned}
\lim_{x\to\infty} \frac{b\ln\left(1+ax^{-1}\right)}{x^{-1}} &\overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{\dfrac{-abx^{-2}}{1+ax^{-1}}}{-x^{-2}} \\
&= \lim_{x\to\infty} \frac{ab}{1+ax^{-1}} \\
\lim_{x\to\infty} \ln y &= ab \qquad \text{[from \#2 label } y\text{]}.
\end{aligned}$$

4. Hence $\lim\limits_{x\to\infty} \left(1 + \dfrac{a}{x}\right)^{bx} = \lim\limits_{x\to\infty} y = \lim\limits_{x\to\infty} e^{\ln y} = e^{ab}$.

MATLAB Examples

s295x70

Find the limit $\lim\limits_{x\to\pi/4} (\tan x)^{\tan 2x}$.

Solution

This is an indeterminate power 1^∞. We simply use MATLAB's **limit** command to dispatch it in one line. Indeed, *any* of the hand examples can be done this way! The limit is $e^{-1} = 1/e$.

```
%-----------------------------------------------------
% Stewart 295/70
%
syms x
y = tan(x) ^ tan(2*x); pretty(y)

                                         tan(2 x)
                                   tan(x)
val = limit(y, x,  sym(pi/4))

val =

exp(-1)


%
echo off; diary off
```