

Fall 2004 Math 151

6 Integrals

6.1 Sigma Notation

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Summary

Sigma notation

For integers $m \leq n$ and real numbers a_m, a_{m+1}, \dots, a_n , we write

$$\sum_{k=m}^n a_k = a_m + a_{m+1} + \dots + a_n.$$

The left-hand side is shorthand for the finite sum on right. The **index of summation** k takes on integer values from m to n . Other indices may be used, such as i and j .

Properties of finite summation

Let c be a constant. Then

- $\sum_{k=m}^n ca_k = c \sum_{k=m}^n a_k$
- $\sum_{k=m}^n (a_k + b_k) = \sum_{k=m}^n a_k + \sum_{k=m}^n b_k$
- $\sum_{k=m}^n (a_k - b_k) = \sum_{k=m}^n a_k - \sum_{k=m}^n b_k$

Particular sums

- (a) $\sum_{k=1}^n 1 = n$
- (b) $\sum_{k=1}^n c = cn$
- (c) $\sum_{k=1}^n k = \frac{n(n+1)}{2}$
- (d) $\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$
- (e) $\sum_{k=1}^n k^3 = \left(\frac{n(n+1)}{2}\right)^2$
- (f) $\sum_{k=1}^n k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

Hand Examples

Apply formulas from the Summary where applicable. These problems are essentially exercises in pattern recognition. As such, they are ripe for computer implementation, as we'll see in the MATLAB examples!

368/2

Write the sum $\sum_{i=1}^6 \frac{1}{i+1}$ in expanded form.

Solution

$$\text{We have } \sum_{i=1}^6 \frac{1}{i+1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}.$$

368/8

Write the sum $\sum_{j=n}^{n+3} j^2$ in expanded form.

Solution

$$\text{We have } \sum_{j=n}^{n+3} j^2 = n^2 + (n+1)^2 + (n+2)^2 + (n+3)^2.$$

368/10

Write the sum $\sum_{i=1}^n f(x_i)\Delta x_i$ in expanded form.

Solution

We have

$$\sum_{i=1}^n f(x_i)\Delta x_i = f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \dots + f(x_n)\Delta x_n.$$

368/14

Write the sum $\frac{3}{7} + \frac{4}{8} + \frac{5}{9} + \frac{6}{10} + \dots + \frac{23}{27}$ in sigma notation.

Solution

$$\text{We have } \sum_{k=3}^{23} \frac{k}{k+4}.$$

368/18

Write the sum $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36}$ in sigma notation.

Solution

This is a sum of reciprocals of squares, $\sum_{k=1}^6 \frac{1}{k^2}$.

369/20

Write the sum $1 - x + x^2 - x^3 + \dots + (-1)^n x^n$ in sigma notation.

Solution

This is an example of an **alternating sum**, $\sum_{k=0}^n (-1)^k x^k$.

369/23

Find the value of the sum $\sum_{j=1}^6 3^{j+1}$.

Solution

We use brute force.

$$\begin{aligned} \sum_{j=1}^6 3^{j+1} &= 3^2 + 3^3 + 3^4 + 3^5 + 3^6 + 3^7 \\ &= 9 + 27 + 81 + 243 + 729 + 2187 \\ &= 3276 \end{aligned}$$

369/26

Find the value of the sum $\sum_{i=1}^{100} 4$.

Solution

We have $\sum_{i=1}^{100} 4 = 4 \sum_{i=1}^{100} 1 = 4(100) = 400$.

369/35

Find the value of the sum $\sum_{i=1}^n (i^3 - i - 2)$.

Solution

Apply the formulas.

$$\begin{aligned} \sum_{i=1}^n (i^3 - i - 2) &= \sum_{i=1}^n i^3 - \sum_{i=1}^n i - 2 \sum_{i=1}^n 1 \\ \text{[Stop; this is fine.] } \rightarrow &= \left(\frac{n(n+1)}{2}\right)^2 - \frac{n(n+1)}{2} - 2n \\ &= \frac{1}{4}(n^4 + 2n^3 + n^2) - \frac{1}{2}(n^2 + n) - 2n \\ \text{[Compare with MATLAB.] } \rightarrow &= \frac{1}{4}n^4 + \frac{1}{2}n^3 - \frac{1}{4}n^2 - \frac{5}{2}n \end{aligned}$$

369/45

Find the limit $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{2}{n} \left(\left(\frac{2i}{n}\right)^3 + 5 \left(\frac{2i}{n}\right) \right)$.

Solution

- First compute the sum. Note that the index of summation is i . The letter n is a *fixed* positive integer. It is i that varies!

$$\begin{aligned} &\sum_{i=1}^n \frac{2}{n} \left(\left(\frac{2i}{n}\right)^3 + 5 \left(\frac{2i}{n}\right) \right) \\ &= \frac{2}{n} \sum_{i=1}^n \left(\frac{2^3}{n^3} i^3 + \frac{10}{n} i \right) \\ &= \frac{2}{n} \left(\frac{2^3}{n^3} \sum_{i=1}^n i^3 + \frac{10}{n} \sum_{i=1}^n i \right) \\ &= \frac{2}{n} \left(\frac{2^3}{n^3} \left(\frac{n(n+1)}{2}\right)^2 + \frac{10}{n} \left(\frac{n(n+1)}{2}\right) \right) \\ &= \frac{2}{n} \left(\frac{2}{n^3} (n^4 + 2n^3 + n^2) + 5(n+1) \right) \\ &= \frac{2}{n} \left(2 \left(n + 2 + \frac{1}{n}\right) + 5n + 5 \right) \\ &= \frac{2}{n} \left(7n + 9 + \frac{2}{n} \right) \\ &= 14 + \frac{18}{n} + \frac{4}{n^2} \end{aligned}$$

- Now take the limit: $\lim_{n \rightarrow \infty} \left(14 + \frac{18}{n} + \frac{4}{n^2} \right) = 14$.
- Therefore, $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{2}{n} \left(\left(\frac{2i}{n}\right)^3 + 5 \left(\frac{2i}{n}\right) \right) = 14$.

If that wasn't a world 'o' hurt, I don't know what is! This is why we use computers: they are better at pattern recognition than you are. See the corresponding MATLAB example.

369/41(c)

Evaluate the **telescoping sum** $\sum_{i=3}^{99} \left(\frac{1}{i} - \frac{1}{i+1} \right)$.

Solution

The nomenclature means that the sum “collapses.”

$$\begin{aligned} \sum_{i=3}^{99} \left(\frac{1}{i} - \frac{1}{i+1} \right) &= \left(\frac{1}{3} - \frac{1}{4} \right) + \left(\frac{1}{4} - \frac{1}{5} \right) + \dots + \left(\frac{1}{98} - \frac{1}{99} \right) + \left(\frac{1}{99} - \frac{1}{100} \right) \\ &= \frac{1}{3} - \frac{1}{100} = \frac{100-3}{300} = \frac{97}{300} \end{aligned}$$

MATLAB Examples

s369x35 [revisited]

Find the value of the sum $\sum_{i=1}^n (i^3 - i - 2)$.

Solution

The MATLAB command **symsum** (symbolic summation) makes quick work of this one! The answer agrees with the one we obtained by hand.

```

%-----
% Stewart 369/35
%
syms i n
our_sum = simplify(symsum(i^3 - i - 2, i, 1, n));
pretty(our_sum)

          4          3          2
1/4 n  + 1/2 n  - 1/4 n  - 5/2 n
%
echo off; diary off

```

s369x45 [revisited]

Find the limit $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{2}{n} \left(\left(\frac{2i}{n} \right)^3 + 5 \left(\frac{2i}{n} \right) \right)$.

Solution

A half page of hand symbolic manipulation is reduced to one line of code. “Can you say power tool?” I knew you could. . .

```

%-----
% Stewart 369/45
%
syms i n
S = symsum(2/n * ( (2*i/n)^3 + 5*(2*i/n) ), i, 1, n);
S = expand(S); pretty(S)

```

$$14 + \frac{18}{n} + \frac{4}{n^2}$$

```

L = limit(S, n, inf)
L =
14
%
echo off; diary off

```

s369x47

Prove the formula for the sum of a **finite geometric series** with first term a and common ratio r .

$$\sum_{i=1}^n ar^{i-1} = a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$$

Solution

```

%-----
% Stewart 369/47: Sum of a finite geometric series
%
syms a i n r
GS = symsum(a * r^(i-1), i, 1, n);
GS = simplify(GS); pretty(GS)

          n
a (r  - 1)
-----
r - 1
%
echo off; diary off

```

s369x48

Evaluate $\sum_{i=1}^n \frac{3}{2^{i-1}}$.

Solution

Rewrite the sum as $\sum_{i=1}^n 3 \left(\frac{1}{2} \right)^{i-1}$, a finite geometric series with $a = 3$ and $r = \frac{1}{2}$. Then apply the result from the preceding

problem to obtain the sum $\frac{3 \left(\left(\frac{1}{2} \right)^n - 1 \right)}{\frac{1}{2} - 1} = 6(1 - 2^{-n})$.

```

%-----
% Stewart 369/48: Sum of a PARTICULAR finite geometric series
%
syms i n
S = simplify( symsum(3 / 2^(i-1), i, 1, n) ); pretty(S)

          (-n)
-6 2  + 6
%
echo off; diary off

```

s369x50

Evaluate $\sum_{i=1}^m \left(\sum_{j=1}^n (i+j) \right)$.

Solution

This is a *double* finite sum.

```
%-----  
% Stewart 369/50: A finite DOUBLE sum  
%  
syms i j m n  
S = symsum( symsum(i+j, j, 1, n), i, 1, m );  
S = simplify(S); pretty(S)  
  
                2          2  
1/2 m n + 1/2 m n + m n  
  
%  
echo off; diary off
```