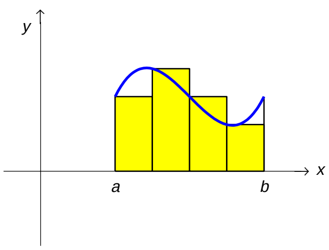
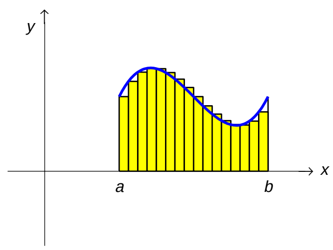
Fall 2004 Math 151
6 Integrals
6.2 Area
Mon, 15/Nov ©2004, Art Belmonte

Summary

Let f be a function defined on $I = [a, b]$ with $f \geq 0$ on I. We seek the area of the region R bounded above by the curve $y = f(x)$, below by the x-axis, on the left by the vertical line $x = a$, and on the right by the vertical line $x = b$. Approximate the area by adding up the areas of rectangular strips as follows.





Split the interval $[a, b]$ into n subintervals whose endpoints constitute a **partition**

$$P : a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

Let $x_i^* \in [x_{i-1}, x_i]$ be in the ith subinterval and $\Delta x_i = x_i - x_{i-1}$ be the length of this subinterval. We define the **norm** of P by $\|P\| = \max_{1 \leq i \leq n} \Delta x_i$. Now let the number of subintervals n increase indefinitely while the norm of P shrinks to 0. The **area** A of R is

$$A = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f\left(x_i^*\right) \Delta x_i,$$

provided that this limit of the sum of the areas of the rectangles formed by the partitions exists.

Hand Examples

Apply formulas from the Section 6.1 Summary when necessary.

377/3

Let $f(x) = 16 - x^2$, $[a, b] = [0, 4]$, $P = \{0, 1, 2, 3, 4\}$, and $x_i^* =$ midpoint.

(a) Find $\|P\|$, the norm of P.

(b) Find $\sum_{i=1}^{n} f\left(x_i^*\right) \Delta x_i$, the sum of the areas of approximating rectangles as given in the Summary.

(c) Sketch the graph of f and the approximating rectangles.

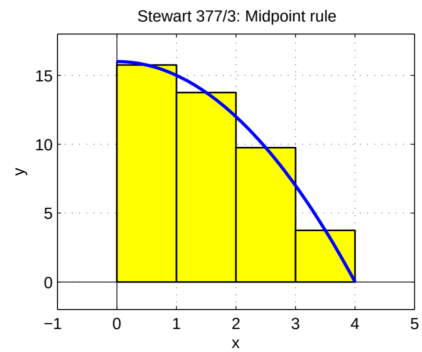
Solution

(a) We have $\|P\| = \max\{1, 1, 1, 1\} = 1$, the length of the longest subinterval in the partition.

(b) The sum of the areas of approximating rectangles is

$$\begin{aligned} & \sum_{i=1}^{4} f\left(x_i^*\right) \Delta x_i \\ = \; & (15.75)(1) + (13.75)(1) + (9.75)(1) + (3.75)(1) \\ = \; & 43. \end{aligned}$$

(c) Here is a graph of f and the approximating rectangles.



377/8

Let $f(x) = 4\cos x$, $[a, b] = \left[0, \frac{\pi}{2}\right]$, $P = \left\{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\right\}$, and $x_i^* =$ left endpoint.

(a) Find $\|P\|$, the norm of P.

(b) Find $\sum_{i=1}^{n} f\left(x_i^*\right) \Delta x_i$, the sum of the areas of approximating rectangles as given in the Summary.

(c) Sketch the graph of f and the approximating rectangles.

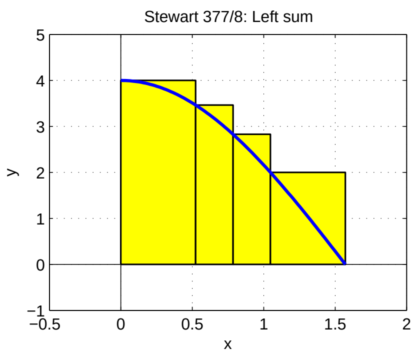
Solution

(a) We have $\|P\| = \max\left\{\frac{\pi}{6}, \frac{\pi}{12}, \frac{\pi}{12}, \frac{\pi}{6}\right\} = \frac{\pi}{6}$, the length of the longest subinterval in the partition.

(b) The sum of the areas of approximating rectangles is

$$\begin{aligned} & \sum_{i=1}^{4} f\left(x_i^*\right) \Delta x_i \\ = \; & (4)\left(\tfrac{\pi}{6}\right) + \left(2\sqrt{3}\right)\left(\tfrac{\pi}{12}\right) + \left(2\sqrt{2}\right)\left(\tfrac{\pi}{12}\right) + (2)\left(\tfrac{\pi}{6}\right) \\ = \; & \tfrac{1}{6}\left(\sqrt{3} + \sqrt{2} + 6\right)\pi \approx 4.789. \end{aligned}$$

(c) Here is a graph of f and the approximating rectangles.

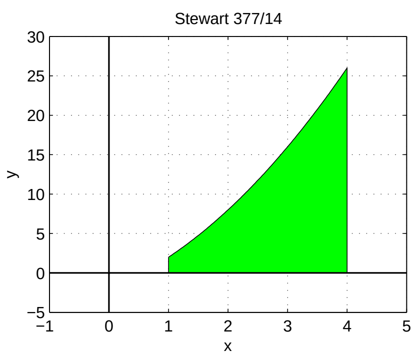


377/14

Find the exact area under the curve $y = f(x) = x^2 + 3x - 2$ and above the x-axis between $a = 1$ and $b = 4$. Use *equal* subintervals and x_k to be the right endpoint of the kth subinterval. Also sketch the region. (NOTE: For brevity, we'll write $\sum$ for $\sum_{k=1}^{n}$).

Solution

- Here is a sketch of the region whose area we seek.



- The length of each subinterval is

$$\Delta x_k = \Delta x = \frac{b-a}{n} = \frac{4-1}{n} = \frac{3}{n}$$

whereas the right endpoint of the kth subinterval is

$$x_k^* = a + k\Delta x = 1 + \frac{3k}{n}.$$

- The sum of the areas of the approximating rectangles is

$$\begin{aligned}
\sum f(x_k^*)\Delta x &= \Delta x \sum f(x_k^*) \quad \text{[since } \Delta x \text{ is constant]} \\
&= \tfrac{3}{n}\sum\left((1+\tfrac{3k}{n})^2 + 3(1+\tfrac{3k}{n}) - 2\right) \\
&= \tfrac{3}{n}\sum\left(1 + \tfrac{6}{n}k + \tfrac{9}{n^2}k^2 + 3 + \tfrac{9}{n}k - 2\right) \\
&= \tfrac{3}{n}\left((\sum 2) + (\tfrac{15}{n}\sum k) + (\tfrac{9}{n^2}\sum k^2)\right) \\
&= \tfrac{3}{n}\left(2n + \tfrac{15n(n+1)}{2n} + \tfrac{9n(n+1)(2n+1)}{6n^2}\right)
\end{aligned}$$

$$= \quad 6 + \frac{45}{2}\left(1+\frac{1}{n}\right) + \frac{9}{2}(1)\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) = S_n$$

- Now let $n \to \infty$ to obtain

$$A = \lim_{n\to\infty} S_n = 6 + \frac{45}{2} + 9 = 15 + \frac{45}{2} = \frac{75}{2} = 37.5.$$

The area is 37.5 square units. (Also see MATLAB example.)

MATLAB Examples

s377x10

Let $f(x) = 4x - x^3$.

(a) Sketch the region that lines under the curve $y = f(x)$ above the x-axis from $x = 0$ to $x = 2$.

(b) Find an expression for R_n, the sum of the areas of the n approximating rectangles, taking x_k^* to be the right endpoint and using subintervals of equal length.

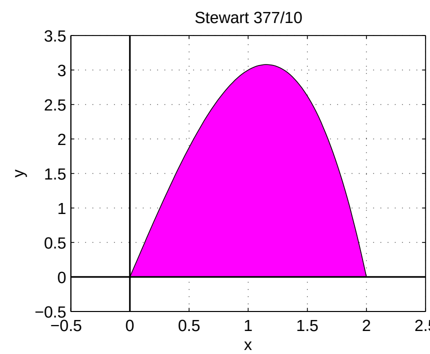
(c) Find the numerical values of the approximating areas R_n for $n = 10, 20, 30$.

(d) Find the exact area of the region.

Solution

A diary file at the end shows all computations and plot commands.

(a) Here is a sketch of the region whose area we seek.



(b) We have $R_n = 4 - \dfrac{4}{n^2}$.

(c) Here are values of R_n for the requested values of n.

n	10	20	30
R_n	3.96	3.99	3.996

(d) The exact area is $\lim\limits_{n\to\infty} R_n = \lim\limits_{n\to\infty}\left(4 - \dfrac{4}{n^2}\right) = 4.$

```
%--------------------------------------------------
% Stewart 377/10: Area of region as limit of sum
% of areas of approximating rectangles
%
% (b)
syms k n x
f = inline('4.*x - x.^3', 'x');
a = 0; b = 2; dx = (b-a)/n; % dx = step size
xk = a + k*dx; % right endpoint of kth subinterval
Rn = symsum(f(xk)*dx, k, 1, n); % right sum
Rn = expand(simplify(Rn)); pretty(Rn)

                                      4
                                 4 - ----
                                       2
                                      n
% (c)
N = [10 20 30]; RSN = [];
for m = N
    RSN = [RSN subs(Rn, n, m)];
    echo off
end
n  Rn
10 3.960000
```

```
20 3.990000
30 3.995556
% echo on
% (d)
A = limit(Rn, n, inf) % area

A =

4


%
echo off; diary off
%--------------------------------------------------
% Stewart 377/10g: Sketch of region
%
f = inline('4.*x - x.^3', 'x');
x = linspace(0, 2);
y = f(x);
xf = [x 2 0]; yf = [y 0 0];
fill(xf,yf, 'm'); hold on
plot([-0.5 2.5], [0 0], 'k', 'LineWidth', 1)
plot([0 0], [-0.5 3.5], 'k', 'LineWidth', 1)
grid on; axis([-0.5 2.5 -0.5 3.5])
xlabel('x'); ylabel('y'); title('Stewart 377/10')
%
echo off; diary off
```

s377x14 [revisited]

Find the exact area under the curve $y = f(x) = x^2 + 3x - 2$ and above the x-axis between $a = 1$ and $b = 4$. Use *equal* subintervals and x_k to be the right endpoint of the kth subinterval.

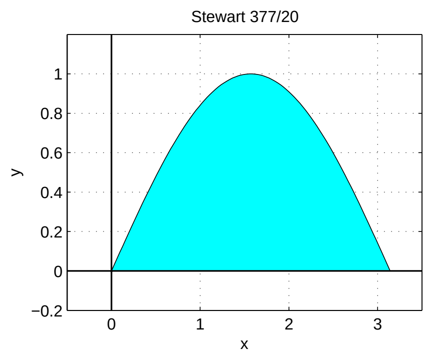
Solution

MATLAB's **symsum** command rapidly yields the needful.

```
%--------------------------------------------------
% Stewart 377/14: Area of region as limit of sum
% of areas of approximating rectangles
%
syms k n x
f = inline('x.^2 + 3.*x - 2', 'x');
a = 1; b = 4; dx = (b-a)/n; % dx = step size
xk = a + k*dx; % right endpoint of kth subinterval
RS = symsum(f(xk)*dx, k, 1, n); % right sum
RS = expand(simplify(RS)); pretty(RS)

                                 36         1
                      75/2 + ---- + 9/2 ----
                                 n          2
                                           n
A = limit(RS, n, inf) % area

A =

75/2


%
echo off; diary off
```

s377x20

Consider the region below the curve $y = f(x) = \sin x$ above the x-axis between $x = 0$ and $x = \pi$. Compute the sum of the areas of approximating rectangles using equal subintervals and right endpoints for $n = 10, 30, 50$. Guess the exact value of the area.



Solution

- Here are values of R_n for the requested values of n.

n	10	30	50
R_n	1.983524	1.998172	1.999342

- The exact area is $\lim_{n\to\infty} R_n = 2$.

```
%--------------------------------------------------
% Stewart 377/20: Area of region as limit of sum
% of areas of approximating rectangles
%
% (b)
syms k n x
f = inline('sin(x)', 'x');
a = 0; b = pi; dx = (b-a)/n; % dx = step size
xk = a + k*dx; % right endpoint of kth subinterval
Rn = symsum(f(xk)*dx, k, 1, n); % right sum
Rn = expand(simplify(Rn)); pretty(Rn)

                                          pi
                                  pi sin(----)
                                          n
                              - -----------------
                                  /     pi      \
                                n |cos(----) - 1|
                                  \     n       /
% (c)
N = [10 30 50]; RSN = [];
for m = N
    RSN = [RSN subs(Rn, n, m)];
    echo off
end
n  Rn
10 1.983524
30 1.998172
50 1.999342
% echo on
% (d)
A = limit(Rn, n, inf) % area

A =

2


%
echo off; diary off
```