4.2 The Mean Value Theorem

1. [292/6] Find all numbers \( c \) that satisfy the conclusion of Rolle’s Theorem for the function \( f(x) = x^3 - 2x^2 - 4x + 2 \) on \([-2, 2]\).

As a polynomial, \( f \) is continuous and differentiable everywhere. Also \( f\{-2, 2\} = \{-6, -6\}. \) So the hypotheses of Rolle’s Theorem are satisfied. Now \( f'(x) = 3x^2 - 4x - 4 = (x - 2)(3x + 2) = 0 \) yields \( c = x = \frac{-2}{3} \in (-2, 2) \) for which \( f'(c) = 0 \), as is guaranteed by the theorem.

2. [292/8] Find all numbers \( c \) that satisfy the conclusion of Rolle’s Theorem for the function \( f(x) = x + 1/x \) on \([\frac{1}{2}, 2]\).

Now \( f \) is continuous and differentiable for \( x \neq 0 \). Moreover, \( f\{\frac{1}{2}, 2\} = \{\frac{1}{2}, \frac{3}{2}\} \). The hypotheses are satisfied. Next, \( f'(x) = 1 - \frac{1}{x^2} = 0 \) gives \( c = x = 1 \in (\frac{1}{2}, 2) \) for which \( f'(c) = 0 \), as guaranteed by the theorem.

3. [292/13] Find all numbers \( c \) that satisfy the conclusion of the Mean Value Theorem for the function \( f(x) = \ln x \) on \([1, 4]\).

Now \( f \) is continuous and differentiable \((0, \infty)\). Hypotheses of the Mean Value Theorem are satisfied. So there’s a number \( c \in (1, 4) \) such that \( f'(c) = \frac{f(4) - f(1)}{4 - 1} = \frac{\ln 4}{3} \). We determine \( c = \frac{3}{\ln 4} \approx 2.16 \in (1, 4) \).

4. [292/14] Find all numbers \( c \) that satisfy the conclusion of the Mean Value Theorem for the function \( f(x) = 1/x \) on \([1, 3]\).

So \( f \) is continuous and differentiable \((0, \infty)\). Hypotheses of the Mean Value Theorem are satisfied. So there’s a \( c \in (1, 3) \) such that \( f'(c) = -\frac{1}{c^2} = \frac{f(3) - f(1)}{3 - 1} = \frac{-\frac{2}{3}}{2} = -\frac{1}{3} \). Hence \( c = \sqrt{3} \approx 1.73 \in (1, 3) \).

5. [292/15] Find all numbers \( c \) that satisfy the conclusion of the Mean Value Theorem for the function \( f(x) = \sqrt{x} \) on \([0, 4]\).

Illustrate graphically by graphing the relevant curve, tangent line, and secant line.

So \( f \) is continuous on \([0, \infty)\) and differentiable on \((0, \infty)\). Hypotheses satisfied. Solve \( f'(c) = \frac{1}{2\sqrt{c}} = \frac{f(4) - f(0)}{4 - 0} = \frac{2}{4} = \frac{1}{2} \) to get \( c = 1 \in (0, 4) \). Here’s a plot.

6. [292/16] Find all numbers \( c \) that satisfy the conclusion of the Mean Value Theorem for the function \( f(x) = e^{-x} \) on \([0, 2]\).

Illustrate graphically by graphing the relevant curve, tangent line, and secant line.

So \( f \) is continuous and differentiable on \( \mathbb{R} \). Hypotheses satisfied. Solve \( f'(c) = -e^{-c} = \frac{f(2) - f(0)}{2 - 0} = \frac{e^{-2} - 1}{2} \) to get \( c = 2 + \ln 2 - \ln (e^2 - 1) \approx 0.84 \in (0, 2) \). Here’s a plot.

7. [292/20] Show that the equation \( x^3 + e^x = 0 \) has exactly one root. Explain why.

Now \( f(x) = x^3 + e^x \) is continuous and differentiable on \( \mathbb{R} \).

Note that \( \lim_{x \to -\infty} f(x) = -\infty \) and \( \lim_{x \to \infty} f(x) = \infty \). So the graph of \( f \) must cross the \( x \)-axis somewhere. That is, the equation has a root. Moreover, \( f'(x) = 3x^2 + e^x > 0 \) for all \( x \).

That is, \( f \) is always increasing. Hence it can only cross the \( x \)-axis once. Therefore, the equation has exactly one root.

8. [292/22] Show that the equation \( x^4 + 4x + c = 0 \) has at most two real roots. Explain why.

Let \( f(x) = x^4 + 4x + c \). Now \( f'(x) = 4x^3 + 4 = 0 \) implies \( x = -1 \) is the only critical value of \( f \). Since \( f' < 0 \) for \( x < -1 \) and \( f' > 0 \) for \( x > -1 \), each member of the family of curves has a local (and absolute) minimum, \( f(-1) = c - 3 \).

So \( f(x) = 0 \) has 2, 1, or 0 roots according to whether \( c < 3 \), \( c = 3 \), or \( c > 3 \), respectively. The picture tells the story.
9. **[292/26]** Suppose that \(3 \leq f'(x) \leq 5\) for all values of \(x\). Show that \(18 \leq f(8) - f(2) \leq 30\).

By the Mean Value Theorem, there is a value \(c\) in \((2, 8)\) such that \(f'(c) = \frac{f(8) - f(2)}{8 - 2} = \frac{f(8) - f(2)}{6}\). By hypothesis, \(3 \leq f'(c) \leq 5\). Thus \(3 \leq \frac{f(8) - f(2)}{6} \leq 5\), and therefore \(18 \leq f(8) - f(2) \leq 30\).

10. **[292/34]** Use the method of Example 6 in your textbook to prove the identity \(2\sin^{-1} x = \cos^{-1} (1 - 2x^2), x \geq 0\).

Let \(f(x) = 2\sin^{-1} x - \cos^{-1} (1 - 2x^2)\). The domain of \(f\) is \([-1, 1]\), on which it is continuous. Note that \(f(0) = 2\sin^{-1} 0 - \cos^{-1} 1 = 2(0) - 0 = 0\).

Let's compute the derivative of \(f\) on \((-1, 0) \cup (0, 1)\).

\[
f'(x) = \frac{2}{\sqrt{1 - x^2}} + \frac{-4x}{\sqrt{1 - (1 - 2x^2)^2}}
\]

\[
= \frac{2}{\sqrt{1 - x^2}} - \frac{4x}{\sqrt{4x^2 - 4x^4}}
\]

\[
= \frac{2}{\sqrt{1 - x^2}} - \frac{2|x|\sqrt{1 - x^2}}{2\sqrt{1 - x^2}}
\]

\[
= \frac{2}{\sqrt{1 - x^2}} - \frac{2\text{sign}(x)}{\sqrt{1 - x^2}}
\]

where \(\text{sign}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}\). Therefore, \(f'(x) = 0\) for \(0 < x < 1\). Hence \(f\) is constant on \((0, 1)\). To determine this constant, \(f\left(\frac{\sqrt{2}}{2}\right) = 2\sin^{-1} \left(\frac{\sqrt{2}}{2}\right) - \cos^{-1} 0 = 2 \left(\frac{\pi}{4}\right) - \frac{\pi}{2} = 0\) will do. So \(f(x) = 0\) on \([0, 1]\). What about \(f'(1)\)? Well since \(f\) is continuous on \([-1, 1]\), \(f(1) = \lim_{x \to 1^-} f(x) = 0\).

We conclude that \(f(x) = 0\) on \([0, 1]\) and thus the identity \(2\sin^{-1} x = \cos^{-1} (1 - 2x^2), x \geq 0\), is proven. (Note that this really means for \(0 \leq x \leq 1\) given the domain of \(f\).)

Here is a case where one glance at a graph will convince you that the identity is true. Below is the graph of \(f\). Clearly over the interval \([0, 1]\) we have \(f(x) = 0\).