1. Evaluate \( \int_0^1 (x+1)^2 e^x \, dx \).

- Use integration by parts. Let \( u = (x+1)^2 \) and \( dv = e^x \, dx \). Then \( du = 2(x+1) \, dx \) and \( v = e^x \).
- Hence
  \[
  \int (x+1)^2 e^x \, dx = (x+1)^2 e^x + \int -2(x+1) e^x \, dx.
  \]
- Reload! Let \( u = -2(x+1) \) and \( dv = e^x \, dx \). Then \( du = -2 \, dx \) and \( v = e^x \).
- Thus
  \[
  \int (x+1)^2 e^x \, dx = (x+1)^2 e^x - 2(x+1) e^x + \int 2 e^x \, dx
  \]
  or
  \[
  \left( (x+1)^2 - 2(x+1) + 2 \right) e^x, \text{ which simplifies upon expansion to } (x^2 + 1) e^x.
  \]
- Therefore,
  \[
  \int_0^1 (x+1)^2 e^x \, dx = \left. (x^2 + 1) e^x \right|_0^1 = 2e - 1 \approx 4.44.
  \]

2. Compute the indefinite integral \( \int \sin^3 x \cos^5 x \, dx \).

- Use the trigonometric identity \( \sin^2 x + \cos^2 x = 1 \).
  \[
  \int \sin^3 x (\cos^2 x)^3 \cos x \, dx
  = \int \sin^5 x (1 - \sin^2 x)^3 \cos x \, dx
  = \int \left( \sin^5 x - 2 \sin^7 x + \sin^9 x \right) \cos x \, dx
  \]
- Now use the Substitution Rule with \( w = \sin x \) and \( dw = \cos x \, dx \).
  \[
  \int w^5 - 2w^7 + w^9 \, dw
  = \frac{1}{6} w^6 - \frac{1}{4} w^8 + \frac{1}{10} w^{10} + C
  = \frac{1}{6} \sin^6 x - \frac{1}{4} \sin^8 x + \frac{1}{10} \sin^{10} x + C
  \]

3. Calculate \( \int_{1/2}^1 \sqrt{1 - \frac{x^2}{3^2}} \, dx \).

- Use a trigonometric substitution. Let \( x = \sin \theta \). Then \( dx = \cos \theta \, d\theta \).
- When \( x = 1/2 \) we have \( \theta = \pi/6 \); whereas when \( x = 1 \) we have \( \theta = \pi/2 \).
- Transform and dispatch the integral.
  \[
  \int_{\pi/6}^{\pi/2} \cos \theta \cdot \cos \theta \, d\theta
  = \int_{\pi/6}^{\pi/2} \cot^2 \theta \, d\theta
  = \int_{\pi/6}^{\pi/2} \csc^2 \theta - 1 \, d\theta
  = \left( - \cot \theta - \theta \right)_{\pi/6}^{\pi/2}
  = \left( - 1 - \frac{\pi}{6} \right) - \left( - \sqrt{3/2} - \frac{\pi}{6} \right)
  = \sqrt{3} - \frac{\pi}{3} \approx 0.68
  \]
  Here we used the fact that \( \cot \theta = \frac{\cos \theta}{\sin \theta} \) and the identity \( \cot^2 \theta = \csc^2 \theta - 1 \). (Just divide the identity \( \sin^2 \theta + \cos^2 \theta = 1 \) by \( \sin^2 \theta \) and rearrange.)

4. Evaluate \( \int \frac{3x^2 + 4x + 4}{x^3 + x} \, dx \).

- Use partial fractions.
  \[
  \frac{3x^2 + 4x + 4}{x(x^2 + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}
  \]
  \[
  3x^2 + 4x + 4 = A(x^2 + 1) + (Bx + C)x
  \]
  \[
  3x^2 + 4x + 4 = (A + B)x^2 + Cx + A
  \]
- Equating like coefficients, we have \( A = 4, C = 4, \) and \( A + B = 3 \), whence \( B = 3 - A = -1 \).
- Now integrate term-by-term.
  \[
  \int \frac{3x^2 + 4x + 4}{x^3 + x} \, dx
  = \int \frac{4}{x} - \frac{x}{x^2 + 1} + \frac{4}{x^2 + 1} \, dx
  = 4 \ln |x| - \frac{1}{2} \ln (x^2 + 1) + 4 \tan^{-1} x + C
  \]
  This is equivalent to \( \ln \frac{x^2}{\sqrt{x^2 + 1}} + 4 \tan^{-1} x + C \).
5. Consider the region in the first quadrant bounded below by the $x$-axis ($y = 0$), on the right by the vertical line $x = 2$, and above by the curve $y = f(x) = e^x \tan^{-1} x \sqrt{1 + x^2}$.

Set up integrals that represent the following quantities. Then calculate them using the m.sum1 TAMUCALC command on your calculator.

(a) Use Simpson’s Rule with $n = 6$ to approximate the volume obtained by rotating the region about the $x$-axis.

- Cross sections are circular disks. The volume is
  \[ V = \int_0^2 \pi r^2 \, dx = \int_0^2 \pi y^2 \, dx \]
  \[ = \int_0^2 \pi \left( e^x \tan^{-1} x \sqrt{1 + x^2} \right)^2 \, dx \]
  \[ \approx 16.70 \text{ cm}^3. \]

(b) Use Simpson’s Rule with $n = 6$ to approximate the volume obtained by rotating the region about the $y$-axis.

- Use cylindrical shells. The volume is
  \[ V = \int 2\pi rh \, dx = \int 2\pi xy \, dx \]
  \[ = \int_0^2 2\pi x \left( e^x \tan^{-1} x \sqrt{1 + x^2} \right) \, dx \]
  \[ \approx 23.64 \text{ cm}^3. \]

6. Here are two items that involve improper integrals.

(a) Use a comparison test to determine whether the integral $\int_1^\infty \frac{1}{\sqrt{e^x + x^3}} \, dx$ is convergent or divergent.

- On $[1, \infty)$, the integrand is positive and less than $\frac{1}{\sqrt{e^x}} = e^{-x/2}$.

- We have
  \[ \int_1^\infty e^{-x/2} \, dx = \lim_{b \to \infty} \int_1^b e^{-x/2} \, dx \]
  \[ = \lim_{b \to \infty} \left( -2e^{-x/2} \right) \bigg|_1^b \]
  \[ = \lim_{b \to \infty} \left( -2e^{-b/2} + 2e^{-1/2} \right) \]
  \[ = 2e^{-1/2} = 2/\sqrt{e} \approx 1.21. \]

- By the Comparison Theorem, the integral converges (to a value less than $2/\sqrt{e} \approx 1.21$). Indeed, the value of the integral is approximately equal to 1.11.

(b) Find the area of the region in the first quadrant that lies between the curves $y = \sec x$ and $y = \tan x$ from $x = 0$ to $x = \pi/2$.

- In the first quadrant, we have
  \[ \sec x = \frac{1}{\cos x} > \sin x = \frac{\sin x}{\cos x} = \tan x. \]

- The area is
  \[ A = \int_0^{\pi/2} \sec x - \tan x \, dx \]
  \[ = \lim_{b \to (\pi/2)^-} \int_0^b \sec x - \frac{\sin x}{\cos x} \, dx \]
  \[ = \lim_{b \to (\pi/2)^-} \left( \ln \left| \sec x + \tan x \right| + \ln \left| \cos x \right| \right) \bigg|_0^b \]
  \[ = \lim_{b \to (\pi/2)^-} \left( \ln \left| 1 + \sin b \right| - 0 \right) \]
  \[ = \ln 2 \approx 0.69 \text{ cm}^2. \]

7. BONUS. Evaluate $\int \sin^2 3x \, dx$.

- Use the trig identity $\sin^2 \theta = \frac{1}{2} (1 - \cos 2\theta)$.
  \[ \int \sin^2 3x \, dx = \frac{1}{2} \int 1 - \cos 6x \, dx \]
  \[ = \frac{1}{2} \left( x - \frac{1}{6} \sin 6x \right) + C \]

8. BONUS. Use the Midpoint Rule with $n = 10$ to approximate the integral $\int_0^1 \sqrt{1 + x^4} \, dx$. Then give an error estimate for the accuracy of your approximation using $|E_M| = \frac{K (b - a)^3}{24n^2}$, where $K = \max_{a \leq x \leq b} |f''(x)|$ and $f(x) = \sqrt{1 + x^4}$. NOTE: For this problem, it turns out that $K = f''(b)$.

- The Midpoint Rule yields $M_{10} \approx 1.08884$.

- Now $f''(x) = \frac{2x^2 (x^4 + 3)}{(x^4 + 1)^{3/2}}$, whence $f''(1) = 2\sqrt{2}$.

- Thus
  \[ |E_M| = \frac{K (b - a)^3}{24n^2} = \frac{2\sqrt{2} (1 - 0)^3}{24(10)^2} = \frac{\sqrt{2}}{1200} \approx 1.18 \times 10^{-3}. \]