1. Consider the sequence $a_1 = \sqrt{6}, a_{n+1} = \sqrt{6 + a_n}, n \geq 1$. Given that the sequence converges (it does), find its limit $L$.

- As $n \to \infty$, we have $a_n, a_{n+1} \to L$. Therefore,
  
  \[ L = \sqrt{6 + L} \geq 0 \]

  \[ L^2 = 6 + L \]

  \[ L^2 - L - 6 = 0 \]

  \[ (L + 2)(L - 3) = 0 \]

  \[ L = -2, 3, \]

  whence $L = 3$ since $L \geq 0$.

2. Find the sum of the series $\sum_{n=1}^{\infty} \left( \frac{17}{3} \left( \frac{1}{2} \right)^{n-1} \right)$.

- This is a geometric series with $|r| = \frac{1}{2} < 1$. By the Geometric Series Theorem (GST), it converges to
  
  \[ a \over 1 - r = \frac{17/3}{1 - (-1/2)} = \frac{17/3}{3/2} = \frac{34}{9} \]

3. Find the sum of the series $\sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+2} \right)$.

- This is a telescoping series. Write down partial sums until you see a pattern.

  \[ s_1 = \left( \frac{1}{1} - \frac{1}{3} \right) = \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \]

  \[ s_2 = \left( \frac{1}{1} - \frac{1}{3} \right) + \left( \frac{1}{2} - \frac{1}{4} \right) = \frac{2}{4} - \frac{1}{4} = \frac{1}{4} \]

  \[ s_3 = \left( \frac{1}{1} - \frac{1}{3} \right) + \left( \frac{1}{2} - \frac{1}{4} \right) + \left( \frac{1}{3} - \frac{1}{5} \right) = \frac{3}{5} - \frac{1}{5} = \frac{2}{5} \]

  \[ \vdots \]

  \[ s_n = \frac{n}{n+1} - \frac{1}{n+2} \]

- The sum of the series is $\lim_{n \to \infty} s_n = 3/2$. The series is convergent.

4. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{(n+1) \sqrt{\ln(n+1)}}$ converges or diverges.

- Since

  \[ \int_1^\infty \frac{1}{(x+1) \sqrt{\ln(x+1)}} \, dx = \lim_{b \to \infty} \left( 2\sqrt{\ln(b+1) - 2\ln 2} \right) = \infty, \]

  this positive series diverges to $\infty$ by the Integral Test.

5. Determine whether the series $\sum_{n=1}^{\infty} \frac{3^n}{4^n + 5}$ converges or diverges.

- The terms of this positive series satisfy

  \[ \frac{3^n}{4^n + 5} < \frac{3^n}{4^n} = \left( \frac{3}{4} \right)^n. \]

  Since the geometric series $\sum_{n=1}^{\infty} \left( \frac{3}{4} \right)^n$ converges ($|r| = \frac{3}{4} < 1$), we conclude that our series converges by the Comparison Test.

6. Determine whether the series $\sum_{n=1}^{\infty} \sin \left( \frac{1}{n} \right)$ converges or diverges. (HINT: Recall the limit $\lim_{\theta \to 0} \frac{\sin \theta}{\theta}$ from Calc 1.)

- Notice that our series is a positive series. We let $\theta = 1/n$. Since the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges ($p$-series with $p = 1 \leq 1$) and $\lim_{n \to \infty} \frac{\sin(1/n)}{1/n} = \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$, we conclude by the Limit Comparison Test that our series diverges.

7. Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{2n+5}$ converges or diverges.

- As $n$ gets large, the terms of the series cluster near $\frac{1}{2}$ and $\frac{1}{2}$. In other words, $\lim_{n \to \infty} a_n = 0$. Accordingly, this series diverges by the Test for Divergence.

8. Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n}$ converges or diverges. If it converges, determine the number of terms required to approximate its sum to within $\varepsilon = 10^{-3}$; then compute this approximation.

- This is an alternating series, the magnitude of whose terms decreases to zero in the limit $b_n = \frac{1}{2^n} \to 0$. To see this, observe that for $n > 1$ we have this reversible chain of inequalities.

  \[ \frac{n+1}{2^{n+1}} < \frac{n}{2^n} \]

  \[ 2^n(n+1) < 2^{n+1}n \]

  \[ n+1 < 2n \]

  \[ 1 < n \]

Hence $|a_n|$ is decreasing after the first term.

Moreover, $\lim_{n \to \infty} \frac{1}{2n \ln 2} = 0$ by L’Hospital’s Rule. We conclude by the Alternating Series Test that our series converges.

- Next, the Alternating Series Estimation Theorem tells us that the remainder $R_n = s - s_n$ satisfies

  \[ |R_n| \leq |a_{n+1}| \leq \frac{\varepsilon}{10^{-3}} \]

  \[ \frac{n+1}{2^{n+1}} \leq 10^{-3} \]

  \[ n \geq 12.75. \]

So choose $n = 13$, whence

\[ \sum_{n=1}^{13} \frac{(-1)^n n}{2^n} = \frac{1825}{8192} \approx 0.223. \]

9. Determine whether the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ converges or diverges.

- As $n \to \infty$, we have

  \[ \frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \left( 1 + \frac{1}{n} \right)^n \to e > 1. \]

The series diverges by the Ratio Test.
10. Determine whether the series \( \sum_{n=1}^{\infty} \left( \frac{\ln n}{n} \right)^n \) converges or diverges.
   - The Root Test gives
     \[
     \lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} \frac{\ln n}{n} = \lim_{n \to \infty} \frac{1}{n} = 0 < 1
     \]
     via L'Hôpital's Rule. The series converges.

11. Find the radius \( R \) and interval \( I \) of convergence for the power series \( \sum_{n=1}^{\infty} \frac{(x-2)^n}{6^n \cdot n} \).
   - The Root Test requires
     \[
     \sqrt[n]{|a_n|} = \left| \frac{x-2}{6} \right| < 1
     \]
     whence \( |x-2| < 6 \) or \(-4 < x < 8\). The radius of convergence is \( R = 6 \).
   - At \( x = 8 \): \( \sum \frac{1}{n} \) diverges (p-series with \( p = 1 \leq 1 \)).
   - At \( x = -4 \): \( \sum \frac{(-1)^n}{n} \) converges via the Alternating Series Test since \( b_n = |a_n| = \frac{1}{n} \to 0 \).
   - Hence the interval of convergence is \( I = [-4,8) \) or \(-4 \leq x < 8 \).

12. (a) Find a power series representation for \( \frac{1}{1+x} \) centered at 0 by using the Geometric Series Theorem.

    Determine its radius of convergence \( R \).
    - We have
      \[
      \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-1)^n x^n,
      \]
      provided \( |x| < 1 \), or \(-1 < x < 1 \).
      The radius of convergence is \( R = 1 \).
    (b) Use part (a) to find a power series representation for \( \frac{-1}{(1+x)^2} \).
      - A power series may be differentiated term-by-term within its radius of convergence.
        \[
        \frac{-1}{(1+x)^2} = \frac{d}{dx} \left( \frac{1}{1+x} \right) = \frac{d}{dx} \left( \sum_{n=0}^{\infty} (-1)^n x^n \right) = \sum_{n=0}^{\infty} (-1)^n n x^{n-1} = \sum_{n=1}^{\infty} (-1)^n n x^{n-1}
        \]
      for \(-1 < x < 1 \).

13. (a) Write down the Maclaurin series for \( \cos z \). State its radius of convergence.
    - We have \( \cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} \). The radius of convergence is \( R = \infty \). The series converges for all real \( z \).
    (b) Use part (a) to write down the Maclaurin series for \( \cos \sqrt{x} \).
      - Let \( z = \sqrt{x} \). Then \( \cos \sqrt{x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^n \).
    (c) Use part (b) to approximate \( \int_0^1 \cos \sqrt{x} \, dx \) to within \( e = 10^{-3} \). Compare this approximation with the exact answer returned by your calculator.
      - Integrate term-by-term.
        \[
        \int_0^1 \cos \sqrt{x} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{x^n}{n!}
        \]
        or
        \[
        \begin{align*}
        \int_0^1 \cos \sqrt{x} \, dx & = \int_0^1 \left( 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots \right) \, dx \\
        & = \left[ x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots \right]_0^1 \\
        & = 1 - \frac{1}{6} + \frac{1}{120} - \frac{1}{5040} + \cdots \\
        & \approx 1 - \frac{1}{6} + \frac{1}{120} = \frac{55}{72} = 0.76388,
        \end{align*}
        \]
      via the Alternating Series Estimation Theorem since \( \frac{1}{2880} < 10^{-3} \). The exact answer is
      \[
      2 \cos 1 + 2 \sin 1 \approx 0.76355 \] and the absolute error is approximately \( 3.4 \times 10^{-4} < 10^{-3} \), as expected.

14. Use the definition to find the Taylor series for \( f(x) = \ln x \) centered at \( a = 4 \). Find its radius of convergence \( R \).
    - Compute derivatives until you see a pattern.
      \[
      \begin{align*}
      f(x) & = \ln x \\
      f'(x) & = 1/x = x^{-1} \\
      f''(x) & = -x^{-2} \\
      f'''(x) & = 2x^{-3} \\
      f^{(4)}(x) & = -6x^{-4} \\
      & \vdots \\
      f^{(n)}(x) & = (-1)^n \frac{n!}{n^n} x^{-n}, \quad n \geq 1
      \end{align*}
      \]
    - The Taylor series is
      \[
      f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(4)}{n!} (x-4)^n = \ln 4 + \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n \cdot n} (x-4)^n.
      \]
    - The Root Test requires
      \[
      \sqrt[n]{|a_n|} = \left| \frac{x-4}{4\sqrt{n}} \right| = \frac{|x-4|}{4^{\frac{1}{2}}} < 1,
      \]
      whence \( |x-4| < 4 \) or \( 0 < x < 8 \). The radius of convergence is \( R = 4 \).