1. Evaluate the integral \( \int x^2 \tan^{-1} x \, dx \) by hand.

- Let \( u = \tan^{-1} x \) and \( dv = x^2 \, dx \). Then \( du = \frac{1}{1+x^2} \, dx \) and \( v = \frac{1}{3} x^3 \).
- Via integration by parts,
  \[
  \int x^2 \tan^{-1} x \, dx = \frac{1}{3} x^3 \tan^{-1} x - \frac{1}{3} \int \frac{x^3}{1+x^2} \, dx
  = \frac{1}{3} x^3 \tan^{-1} x - \frac{1}{3} \int x - \frac{x}{1+x^2} \, dx
  = \frac{1}{3} x^3 \tan^{-1} x - \frac{1}{6} x^2 + \frac{1}{6} \ln (1+x^2) + C.
  \]

2. Compute the integral \( \int \tan^3 x \, dx \) by hand.

- Use a trig identity, then integrate term by term.
  \[
  \int \tan x \cdot \tan^2 x \, dx
  = \int \tan x (\sec^2 x - 1) \, dx
  = \int \tan x \sec^2 x - \frac{\sin x}{\cos x} \, dx
  = \frac{1}{2} \tan^2 x + \ln (\cos x) + C
  \]

3. Evaluate the integral \( \int \frac{1}{(25 + 16x^2)^{3/2}} \, dx \) by hand using a trigonometric substitution.

- The form of the substitution is \( u = a \tan \theta \). Thus \( 4x = 5 \tan \theta \) or \( x = \frac{5}{4} \tan \theta \). Hence \( dx = \frac{5}{4} \sec^2 \theta \, d\theta \).
- Substitute and dispatch.
  \[
  \int \frac{\frac{5}{4} \sec^2 \theta}{5^3 \sec^2 \theta} \, d\theta = \frac{1}{100} \int \cos \theta \, d\theta
  = \frac{1}{100} \sin \theta + C
  = \frac{1}{100} \frac{4x}{\sqrt{25 + 16x^2}} + C
  \]
  \[
  \text{or } \frac{x}{25 \sqrt{25 + 16x^2}} + C. \text{ Note } \tan \theta = \frac{4x}{5} = \frac{\text{opp}}{\text{adj}} \implies \sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{4x}{\sqrt{25 + 16x^2}}.
  \]

4. Compute the integral below using `expand` on your calculator to obtain a partial fraction decomposition of the integrand, then integrating term by term.

\[
\int \frac{x^3 + 3x^2 - 4x + 6}{(x-1)^2(x^2+1)} \, dx
\]

- The `expand` command renders the needful, then we take it from there:
  \[
  \int \frac{3x}{2x^2+1} + \frac{5}{2} + \frac{1}{2} x - \frac{1}{2} \ln (x-1) - 3(x-1)^{-1} + C
  = \frac{3}{4} \ln (x^2 + 1) + \frac{5}{2} \tan^{-1} x - \frac{1}{2} \ln (x-1) - 3(x-1)^{-1} + C
  \]

5. Use the substitution \( w = x^{1/3} \) to change the improper integral \( \int_{0}^{\infty} \frac{1}{x^{2/3} + x^{4/3}} \, dx \) into a simpler one. (First factor out \( x^{2/3} \) in the denominator.) Then compute it step by step using limits.

- Factoring out \( x^{2/3} \), we have
  \[
  \int_{0}^{\infty} \frac{1}{x^{2/3} (1 + x^{2/3})} \, dx = \int_{0}^{\infty} \frac{1}{1 + (x^{1/3})^2} x^{-2/3} \, dx.
  \]
- Note that the integral is doubly improper.
  - The interval of integration is infinite.
  - There is an infinite discontinuity of the integrand at \( x = 0 \).

- If \( w = x^{1/3} \), then \( dw = \frac{1}{3} x^{-2/3} \, dx \) or \( 3 \, dw = x^{-2/3} \, dx \). As \( x \to 0^- \), \( w \to 0^+ \); as \( x \to \infty \), \( w \to \infty \). Enter the \( w \)-world!

  \[
  3 \int_{0}^{\infty} \frac{1}{1 + w^2} \, dw
  = \lim_{b \to \infty} 3 \int_{a}^{b} \frac{1}{1 + w^2} \, dw
  = \lim_{b \to \infty} 3 \tan^{-1} w \bigg|_{a}^{b}
  = 3 \lim_{b \to \infty} \left( \tan^{-1} b - \tan^{-1} a \right)
  = 3 \left( \frac{\pi}{2} - 0 \right) = \frac{3}{2} \pi \approx 4.71
  \]

6. Graph the polar curve \( r = 3 \sin 3\theta \), \( 0 \leq \theta \leq \pi \), using your calculator. Draw a sketch below. Now find the approximate arc length of the curve.

The arc length is \( L = \int_{\alpha}^{\beta} \sqrt{r^2 + (dr/d\theta)^2} \, d\theta \)

\[
= \int_{0}^{\pi} \sqrt{(3 \sin 3\theta)^2 + (9 \cos 3\theta)^2} \, d\theta \approx 20.05 \text{ cm}.
\]
7. Find the center \( C \) and radius \( r \) of the sphere with equation \( 3x^2 + 3y^2 + 3z^2 - 18z - 48 = 0 \).

- Divide the equation by 3 so as to make the coefficients in front of the squared terms 1.
  \[ x^2 + y^2 + z^2 - 6z - 16 = 0 \]
- Add 16 to each side; complete the square in \( z \).
  \[ x^2 + y^2 + (z - 3)^2 = 5^2 \]
- The center is \( C(0,0,3) \). The radius is 5.

8. A Canadian boy pulls a sled across the snow using a rope that makes an angle of \( 30^\circ \) or \( \pi/6 \) radians with the sled. He exerts a constant force of 100 N. Find the work done pulling the sled 1 km.

- Recall that 1 km is 1000 m.
- The work done is
  \[
  W = F \cdot D = \|F\| \|D\| \cos \theta
  = (100)(1000) \left( \frac{\sqrt{3}}{2} \right)
  = 50000\sqrt{3} \approx 86602.54 \text{ J}
  \]
  or approximately 86.6 kilojoules.

9. Find the volume of a parallelepiped (sheared box) spanned by the position vectors

\[
\mathbf{P} = \begin{bmatrix} 1 & 7 & 8 \end{bmatrix},
\mathbf{Q} = \begin{bmatrix} -5 & 9 & -2 \end{bmatrix},
\mathbf{R} = \begin{bmatrix} -6 & 8 & 6 \end{bmatrix}.
\]

- The volume equals the absolute value of the scalar triple product of \( \mathbf{P}, \mathbf{Q}, \) and \( \mathbf{R} \).
  \[
  V = |\mathbf{P} \cdot (\mathbf{Q} \times \mathbf{R})|
  = |\begin{bmatrix} 1 & 7 & 8 \end{bmatrix} \cdot \begin{bmatrix} 70 & 42 & 14 \end{bmatrix}|
  = |476| = 476 \text{ cm}^3
  \]