1. Determine the limit $p$ of the recursive sequence

\[ a_1 = 1, \quad a_{n+1} = \frac{1}{3} \left( 2a_n + \frac{27}{a_n^2} \right), \quad n \geq 1. \]

- Let the limit of the sequence be $p$. As $n \to \infty$, the formula tends to $p = \frac{1}{3} \left( 2p + \frac{27}{p^2} \right)$.

Hence $p = 3$ via solve or by hand.

2. Determine whether the geometric series

\[ \sum_{k=0}^{\infty} 2 \left( -\frac{e}{3} \right)^k \]

converges/diverges. If it converges, find its sum via the Geometric Series Theorem. Show your work. (NOTE: $e \approx 2.72$.)

- Now $a = 2$ and $r = -e/3$. Since $|r| = e/3 < 1$, the series converges by the GST to

\[ \frac{a}{1-r} = \frac{2}{1 + \frac{e}{3}} = \frac{6}{3+e} \approx 1.05. \]

3. Let $a_n = \frac{1}{n \cdot (n+2)}$.

(a) Use the expand command to obtain a partial fraction decomposition of $a_n$. We’ll use this in part (b).

- \[ a_n = \frac{1}{2n} - \frac{1}{(n+2)} = \frac{1}{2} \left( \frac{1}{n} - \frac{1}{n+2} \right) \]

(b) From part (a), we see that $\sum_{n=1}^{\infty} \frac{1}{n \cdot (n+2)}$ is a telescoping series. It converges; find its sum.

- Here are the first few partial sums plus the general one. (We factor $\frac{1}{2}$ out of each term to simplify computations.)

\[ \frac{1}{2} \left( 1 - \frac{1}{3} \right) = s_1 = \frac{1}{2} \left( 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} \right) = s_2 = \frac{1}{2} \left( 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} \right) = s_3 = \frac{1}{2} \left( 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} \right) = s_4 = \frac{1}{2} \left( 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right) = s_n \]

- Therefore, the sum of the series is

\[ s = \lim_{n \to \infty} s_n = \frac{1}{2} \left( \frac{3}{2} \right) = \frac{3}{4} = 0.75. \]

4. Determine all values of $p$ for which the series

\[ \sum_{n=1}^{\infty} \frac{n}{n^2 + 1} \]

converges. Consider three cases.

\[ p < 1 \quad p = 1 \quad p > 1 \]

Employ the Integral Test.

- For $p < 1$, we have $q = -p + 1 > 0$. Thus

\[ \int_1^{\infty} (x^2 + 1)^{-p} \, dx = \lim_{b \to \infty} \frac{1}{b} \ln (b^2 + 1) - \ln 2 = \infty. \]

The series diverges by the Integral Test.

- For $p = 1$, 

\[ \int_1^{\infty} \frac{x}{x^2 + 1} \, dx = \lim_{b \to \infty} \frac{1}{2} \ln (b^2 + 1) - \ln 2 = \infty. \]

Again, the series diverges by the Integral Test.

- For $p > 1$, we have $q = -p + 1 < 0$. Hence

\[ \int_1^{\infty} (x^2 + 1)^{-p} \, dx = \lim_{b \to \infty} \frac{1}{b} \ln (b^2 + 1) - \ln 2 = \infty. \]

So the series converges by the Integral Test.

- Hence the series converges for $p > 1$ only.

5. Determine if the series \[ \sum_{n=1}^{\infty} \frac{1}{1 + 2 + \cdots + n} \]

converges.

First use the sum template on your calculator with index $k$ to express the denominator in a closed form. Then apply a relevant test.

\[ \frac{1}{2} \left( 1 + \frac{1}{1 - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} - \frac{1}{6} - \frac{1}{7} - \cdots - \frac{1}{k} - \frac{1}{k+1} - \frac{1}{k+2} \right) \]

The denominator $1 + 2 + \cdots + n$ is

\[ \frac{1}{2} \left( 1 + \frac{1}{1 - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} - \frac{1}{6} - \frac{1}{7} - \cdots - \frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+2} \right) \]

The (positive) series is therefore \[ \sum_{n=1}^{\infty} \frac{2}{n^2 + n} \]

Since \[ \frac{2}{n^2 + n} < \frac{2}{n^2} \]
and \[ 2 \sum_{n=1}^{\infty} \frac{1}{n^2} \]
converges (since it’s a multiple of a convergent $p$-series where $p = 2 > 1$), our positive series converges by the Comparison Test.
6. For each series, determine whether it converges or diverges, citing a relevant test. Show work below.

(a) \[ \sum_{n=1}^{\infty} \frac{(\sin n)^2}{n^2} \]
- Since \( \frac{(\sin n)^2}{n^2} \leq \frac{1}{n^2} \) and \( \sum \frac{1}{n^2} \) is a convergent \( p \)-series \( (p = 2 > 1) \), our positive series converges by the Comparison Test.

(b) \[ \sum_{n=1}^{\infty} \frac{(-1)^n \tan^{-1} n}{n} \]
- This is an alternating series with \( |a_n| = \frac{\tan^{-1} n}{n} \downarrow 0 \). Hence the series converges by the Alternating Series Test.

7. Find the radius of convergence \( R \) of the series \( \sum_{n=1}^{\infty} \frac{n!}{n^n} x^n \). Use the limit template on your calculator.
- Via the Ratio Test,
  \[
  \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+1)!|x|^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{n!|x|^n} \right| = \left| \frac{(n+1)n}{n+1} \cdot \left( \frac{n}{n+1} \right)^n \right| = \left| \frac{n}{n+1} \right|^n |x|
  \]
  (as \( n \to \infty \)) \( \to |x| \) need \( e \) \( < 1 \)
  - Or \( |x| < e \).
  - The radius of convergence is \( R = e \approx 2.72 \).

8. Let \( f(x) = \frac{x+2}{x^2+x+1} \).
(a) Compute \( T_3(x) \), the 3rd degree Taylor polynomial of \( f(x) \) at \( a = 0 \). (Use the taylor command.)
  - The command \textit{taylor}(\frac{x+2}{x^2+x+1}, x, 3) yields \( T_3(x) = 2 - x - x^2 + 2x^3 \).
(b) Compute \( \int_0^{1/2} T_3(x) \, dx \) exactly. Also give a 2-digit decimal approximation. Name it \( b \).
  - The integral equals \( b = \frac{81}{96} \approx 0.86 \).
(c) Compute \( \int_0^{1/2} f(x) \, dx \) exactly. Also give a 2-digit decimal approximation. Name it \( c \).
  - The integral equals \( c = \sqrt{3} \tan^{-1} \left( \frac{2\sqrt{3}}{3} \right) + \frac{1}{2} \ln \left( \frac{7}{4} \right) - \frac{\sqrt{3}}{6} \pi \) or approximately 0.86.
(d) Compute \( |b-c| \), the absolute error.
  - The absolute error is approximately \( 7.18 \times 10^{-3} \).

9. Find the Maclaurin series (i.e., the Taylor series with center \( a = 0 \)) for the function \( f(x) = \sinh (x) \), the hyperbolic sine of \( x \). Also determine its radius of convergence \( R \). Use the \textit{taylor} command to help with the pattern.
  - The command \textit{taylor}(\sinh(x), x, 7) yields \( x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} \).
  - Observing the pattern gives the series \( \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} \).
  - The Ratio Test yields
    \[
    \left| \frac{a_{k+1}}{a_k} \right| = \left| \frac{|x|^{2k+3}}{(2k+3)!} \cdot \frac{(2k+1)!}{|x|^{2k+1}} \right| = \left| \frac{|x|^2}{(2k+2)(2k+3)} \right|
    \]
    (as \( k \to \infty \)) \( \to 0 < 1 \) for all \( x \).
  - The radius of convergence is \( R = \infty \).