Fall 2004 Math 151
6 Integrals
6.4 The Fundamental Theorem of Calculus
Mon, 22/Nov ©2004, Art Belmonte

Summary

Fundamental Theorem of Calculus (FTC)

Let f be a continuous function on $[a, b]$. Then

Part 1 If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.

Part 2 $\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$,
where F is an antiderivative of f; i.e., $F' = f$.

Thus differentiation and integration are seen to be inverse processes. The second part of the theorem makes it *much easier* to compute definite integrals by hand than by evaluating limits of Riemann sums as we did in Section 6.3. In MATLAB, the **int** command can compute a definite integral in one line.

Hand Examples

You'll find the going quite a bit easier now that you have a theoretical power tool in the Fundamental Theorem of Calculus!

398/4

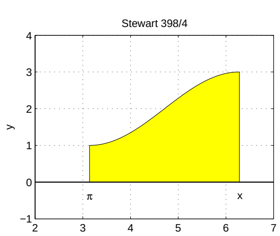
Sketch the area represented by $g(x) = \int_\pi^x 2 + \cos t\,dt$.
Then find $g'(x)$ in two ways:

(a) by using Part 1 of the Fundamental Theorem of Calculus;

(b) by evaluating the definite integral using Part 2 of the Fundamental Theorem of Calculus and then differentiating.

Solution

Here is a sketch of the area represented by the integral.



(a) Part 1 of the FTC gives $g'(x) = 2 + \cos x$.

(b) Applying Part 2 of the FTC, we have

$$\begin{aligned} g(x) &= \int_\pi^x 2 + \cos t\,dt \\ &= (2t + \sin t)\Big|_{t=\pi}^{t=x} \\ &= (2x + \sin x) - (2\pi + \sin\pi) \\ &= 2x + \sin x - 2\pi. \end{aligned}$$

That is, $g(x) = 2x + \sin x - 2\pi$. Then $g'(x) = 2 + \cos x$.

398/6

Use Part 1 of the FTC to find the derivative of

$$g(x) = \int_{-1}^x \sqrt{t^3 + 1}\,dt.$$

Solution

We have $g'(x) = \sqrt{x^3 + 1}$.

398/13

Use Part 1 of the FTC to find the derivative of

$$y = \int_{\tan x}^{17} \sin\left(t^4\right)\,dt.$$

Solution

Let $u = \tan x$. Then

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\,\frac{du}{dx} \\ &= \frac{d}{du}\left(\int_u^{17} \sin\left(t^4\right)\,dt\right)\,\frac{d}{dx}(\tan x) \\ &= \frac{d}{du}\left(-\int_{17}^u \sin\left(t^4\right)\,dt\right)\,\frac{d}{dx}(\tan x) \\ &= -\sin\left(u^4\right)\sec^2 x \\ &= -\sin\left(\tan^4 x\right)\sec^2 x. \end{aligned}$$

398/22

Evaluate the integral $\int_0^1 y^9 - 2y^5 + 3y\,dy$.

Solution

Apply the Fundarmental Theorem of Calculus (FTC). A decimal approximation to the exact answer is provided for comparison.

$$\begin{aligned}\int_0^1 y^9 - 2y^5 + 3y\,dy &= \left(\tfrac{1}{10}y^{10} - \tfrac{1}{3}y^6 + \tfrac{3}{2}y^2\right)\Big|_0^1 \\ &= \left(\tfrac{1}{10} - \tfrac{1}{3} + \tfrac{3}{2}\right) - (0 - 0 + 0) \\ &= \frac{19}{15} \approx 1.27.\end{aligned}$$

398/26

Evaluate the integral $\int_1^2 \frac{t^6 - t^2}{t^4}\,dt$.

Solution

We have

$$\begin{aligned}\int_1^2 \frac{t^6 - t^2}{t^4}\,dt &= \int_1^2 t^2 - t^{-2}\,dt \\ &= \left(\tfrac{1}{3}t^3 + t^{-1}\right)\Big|_1^2 \\ &= \left(\tfrac{8}{3} + \tfrac{1}{2}\right) - \left(\tfrac{1}{3} + 1\right) \\ &= \frac{11}{6} \approx 1.83.\end{aligned}$$

398/28

Evaluate the integral $\int_0^2 \left(x^3 - 1\right)^2\,dx$.

Solution

We have

$$\begin{aligned}\int_0^2 \left(x^3 - 1\right)^2\,dx &= \int_0^2 x^6 - 2x^3 + 1\,dx \\ &= \left(\tfrac{1}{7}x^7 - \tfrac{1}{2}x^4 + x\right)\Big|_0^2 \\ &= \left(\tfrac{128}{7} - 8 + 2\right) - (0) \\ &= \frac{86}{7} \approx 12.29.\end{aligned}$$

398/34

Evaluate the integral $\int_{-1}^2 \left|x - x^2\right|\,dx$.

Solution

Here we must split the interval of integration to resolve the absolute value involved. Note that $|x\,(1-x)| = x\,(1-x)$ for $0 \le x \le 1$ and $x\,(x-1)$ elsewhere. Therefore

$$\begin{aligned}&\int_{-1}^2 \left|x - x^2\right|\,dx \\ =\;& \int_{-1}^0 x^2 - x\,dx + \int_0^1 x - x^2\,dx + \int_1^2 x^2 - x\,dx \\ =\;& \left(\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2\right)\Big|_{-1}^0 + \left(\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right)\Big|_0^1 + \left(\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2\right)\Big|_1^2 \\ =\;& -\left(-\tfrac{1}{3} - \tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) + \left(\tfrac{8}{3} - 2\right) - \left(\tfrac{1}{3} - \tfrac{1}{2}\right) \\ =\;& \frac{11}{6} \approx 1.83.\end{aligned}$$

398/35

Evaluate the integral $\int_{-4}^2 \frac{2}{x^6}\,dx$ or show why it does not exist.

Solution

First off, we *cannot* apply the FTC since the integrand is *not* continuous on $[-4, 2]$. Indeed, it has an infinite discontinuity at $x = 0$. The integral in this case does not exist as a finite real number. (Indeed, we'll see Calc 2 that $\int_{-4}^2 \frac{2}{x^6}\,dx = \infty$. For now, see the correpsonding MATLAB example.)

398/43

Evaluate the integral $\int_{\pi/2}^{\pi} \sec x \tan x\,dx$ or show why it does not exist.

Solution

We *cannot* apply the FTC since the integrand is *not* continuous on $\left[\frac{\pi}{2}, \pi\right]$. Indeed, it has an infinite discontinuity at $x = \frac{\pi}{2}$. The integral in this case does not exist as a finite real number. (Indeed, we'll see Calc 2 that $\int_{\pi/2}^{\pi} \sec x \tan x\,dx = \infty$. For now, see the correpsonding MATLAB example.)

398/45

Evaluate the integral $\int_{\pi/6}^{\pi/3} \csc^2\theta\,d\theta$.

Solution

We have
$$\int_{\pi/6}^{\pi/3} \csc^2\theta\, d\theta = (-\cot\theta)\Big|_{\pi/6}^{\pi/3} = \left(-\tfrac{1}{3}\sqrt{3}\right) - \left(-\sqrt{3}\right) = \tfrac{2}{3}\sqrt{3}$$
or approximately 1.15.

398/48

Evaluate the integral $\int_{\ln 3}^{\ln 6} 8e^x\, dx$.

Solution

We have $\int_{\ln 3}^{\ln 6} 8e^x\, dx = 8e^x\Big|_{\ln 3}^{\ln 6} = 8\,(6) - 8\,(3) = 24.$

398/50

Evaluate the integral $\int_{-e^2}^{-e} \frac{3}{x}\, dx$.

Solution

We have $\int_{-e^2}^{-e} \frac{3}{x}\, dx = 3\ln|x|\,\Big|_{-e^2}^{-e} = 3 - 6 = -3.$

398/52

Evaluate the integral $\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\, dx$.

Solution

We have $\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x\Big|_0^{1/2} = \frac{\pi}{6} - 0 = \frac{\pi}{6} \approx 0.52.$

398/60

Evaluate the integral $\int_{-\pi}^{\pi} f(x)\, dx$ where

$$f(x) = \begin{cases} x & \text{if } -\pi \le x \le 0 \\ \sin x & \text{if } 0 < x \le \pi \end{cases}.$$

Solution

Split the interval of integration.

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x)\, dx &= \int_{-\pi}^{0} f(x)\, dx + \int_0^{\pi} f(x)\, dx \\
&= \int_{-\pi}^{0} x\, dx + \int_0^{\pi} \sin x\, dx \\
&= \left(\tfrac{1}{2}x^2\Big|_{-\pi}^{0}\right) + \left((-\cos x)\Big|_0^{\pi}\right) \\
&= \left(0 - \tfrac{1}{2}\pi^2\right) + (1+1) \\
&= 2 - \tfrac{1}{2}\pi^2 \approx -2.93.
\end{aligned}$$

399/75

Find the general indefinite integral: $\int 2x + \sec x \tan x\, dx$.

Solution

This is another word for antiderivative. Don't forget to add an arbitrary constant!

$$\int 2x + \sec x \tan x\, dx = x^2 + \sec x + C$$

399/78

A particle moves with velocity $v(t) = t^2 - 2t - 8$ (in m/s) along a line (say the x-axis).

(a) Find the displacement (change in position) during the time interval $1 \le t \le 6$.

(b) Find the distance traveled by the particle during the time interval $1 \le t \le 6$.

Solution

(a) Position $x(t)$ is an antiderivative of velocity. Therefore,

$$\begin{aligned}
\text{displacement} &= \int_1^6 v(t)\, dt \\
&= \int_1^6 t^2 - 2t - 8\, dt \\
&= \left(\tfrac{1}{3}t^3 - t^2 - 8t\right)\Big|_1^6 \\
&= -12 - \left(-\tfrac{26}{3}\right) = -\tfrac{10}{3} \approx -3.33 \text{ m}.
\end{aligned}$$

(b) Distance traveled is the integral of speed, $|v(t)|$. Now $v(t) = (t+2)(t-4)$, whence $v(t) \le 0$ on $[1,4]$ and $v(t) \ge 0$ on $[4,6]$. Thus

$$\begin{aligned} \text{distance} &= \int_1^6 |v(t)|\, dt \\ &= \int_1^4 -v(t)\, dt + \int_4^6 v(t)\, dt \\ &= \left(\left(8t + t^2 - \tfrac{1}{3}t^3\right)\Big|_1^4\right) + \left(\left(\tfrac{1}{3}t^3 - t^2 - 8t\right)_4^6\right) \\ &= \left(\tfrac{80}{3} - \tfrac{26}{3}\right) + \left(-12 - \left(-\tfrac{80}{3}\right)\right) \\ &= 18 + \frac{44}{3} = \frac{98}{3} \approx 32.67 \text{ m.} \end{aligned}$$

399/80

A particle moves with acceleration $a(t) = 2t + 3$ (in m/s^2) along a line (say the x-axis). Its initial velocity is $v(0) = -4$.

(a) Find its velocity at time t.

(b) Find the distance traveled during the time interval $0 \le t \le 3$.

Solution

(a) Velocity $v(t)$ is an antiderivative of acceleration. Therefore,

$$\begin{aligned} \text{velocity} &= \int a(t)\, dt \\ \text{velocity} &= \int 2t + 3\, dt \\ v(t) &= t^2 + 3t + C \\ -4 = v(0) &= C \\ v(t) &= t^2 + 3t - 4. \end{aligned}$$

(b) Distance traveled is the integral of speed, $|v(t)|$. Now $v(t) = (t-1)(t+4)$, whence $v(t) \le 0$ on $[0,1]$ and $v(t) \ge 0$ on $[1,3]$. Thus

$$\begin{aligned} \text{distance} &= \int_0^3 |v(t)|\, dt \\ &= \int_0^1 -v(t)\, dt + \int_1^3 v(t)\, dt \\ &= \left(\left(-\tfrac{1}{3}t^3 - \tfrac{3}{2}t^2 + 4t\right)\Big|_0^1\right) + \left(\left(\tfrac{1}{3}t^3 + \tfrac{3}{2}t^2 - 4t\right)_1^3\right) \\ &= \left(4 - \tfrac{11}{6}\right) + \left(\tfrac{21}{2} - \left(-\tfrac{13}{6}\right)\right) \\ &= \frac{13}{6} + \frac{38}{3} = \frac{89}{6} \approx 14.83 \text{ m.} \end{aligned}$$

399/82

An animal population is increasing at a rate of $200 + 50t$ per year, (where t is measured in years). By how much does the animal population increase between the fourth and tenth years?

Solution

The increase is

$$\int_4^{10} 200 + 50t\, dt = \left(200t + 25t^2\right)\Big|_4^{10} = 4500 - 1200 = 3300.$$

399/88

Find the derivative of $y = \int_{\cos x}^{5x} \cos\left(u^2\right)\, du$.

Solution

Let $v = \cos x$ and $w = 5x$. Also, note that

$$\begin{aligned} \int_{\cos x}^{5x} \cos\left(u^2\right)\, du &= \int_{\cos x}^{0} \cos\left(u^2\right)\, du + \int_0^{5x} \cos\left(u^2\right)\, du \\ &= -\int_0^{\cos x} \cos\left(u^2\right)\, du + \int_0^{5x} \cos\left(u^2\right)\, du \\ &= -y_1 + y_2. \end{aligned}$$

Accordingly, we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left(-y_1 + y_2\right) \\ &= -\frac{dy_1}{dx} + \frac{dy_2}{dx} \\ &= -\frac{dy_1}{dv}\frac{dv}{dx} + \frac{dy_2}{dw}\frac{dw}{dx} \\ &= -\frac{d}{dv}\left(\int_0^v \cos\left(u^2\right)\, du\right)\frac{d}{dx}(\cos x) + \frac{d}{dw}\left(\int_0^w \cos\left(u^2\right)\, du\right)\frac{d}{dx}(5x) \\ &= -\cos\left(v^2\right)(-\sin x) + \cos\left(w^2\right)\cdot 5 \\ &= \sin x \cos\left(\cos^2 x\right) + 5\cos\left(25x^2\right). \end{aligned}$$

400/96

Evaluate $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\sqrt{\frac{k}{n}}$ by first recognizing a Riemann sum for a function defined on $[0,1]$.

Solution

Examine the pieces and flesh them out a little.

$$\lim_{n\to\infty} \frac{1-0}{n}\sum_{i=1}^{n}\sqrt{0 + k\left(\frac{1-0}{n}\right)}$$

We recognize this as the limit of the right sums of the integral $\int_0^1 \sqrt{x}\,dx$. Accordingly its value is

$$\int_0^1 x^{1/2}\,dx = \frac{2}{3}x^{3/2}\Big|_0^1 = \frac{2}{3} - 0 = \frac{2}{3} \approx 0.67.$$

400/102

Find a function f such that $f(1) = 0$ and $f'(x) = 2^x/x$.

Solution

By Part 1 of the FTC, the function $f(x) = \int_1^x \frac{2^t}{t}\,dt$ fits the bill, since

$$\begin{aligned} f(1) &= \int_1^1 \frac{2^t}{t}\,dt = 0, \\ f'(x) &= \frac{d}{dx}\left(\int_1^x \frac{2^t}{t}\,dt\right) = \frac{2^x}{x}. \end{aligned}$$

399/90

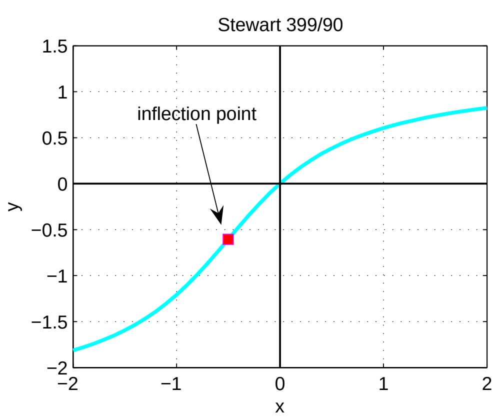
Find the interval on which the curve

$$y = \int_0^x \frac{1}{1+t+t^2}\,dt$$

is concave upward.

Solution

Now $y' = \dfrac{1}{1+x+x^2}$. Thus $y'' = -\dfrac{2x+1}{(1+x+x^2)^2} > 0$ for $x < -\frac{1}{2}$. Hence y is concave up on $\left(-\infty, -\frac{1}{2}\right)$. Here is a plot that corroborates this.



MATLAB Examples

s398x22 [398/22 revisited]

Evaluate the integral $\int_0^1 y^9 - 2y^5 + 3y\,dy$.

Solution

Here are two solutions. To just see the final answer, use **int**.

```
%-------------------------------------------------
% Stewart 398/22: All at once
%
syms y
int(y^9 - 2*y^5 + 3*y, y, 0, 1)

ans =

19/15

%
echo off; diary off
```

To see the steps, use **smi**, "stepwise multiple integration," a page from the Calc 3 playbook!

```
%-------------------------------------------------
% Stewart 398/22: Step-by-step!
%
syms y
smi(y^9 - 2*y^5 + 3*y, [y 0 1]);
STEPWISE (MULTIPLE) INTEGRATION!

Antiderivative w.r.t. y:

                                    10         6          2
                          1/10 y   - 1/3 y   + 3/2 y
When y = 1:

                                             19
                                             --
                                             15
When y = 0:

                                              0
Difference:

                                             19
                                             --
                                             15
Answer (above) and approximaton (below)
    1.2667
%
echo off; diary off
```

s398x35

Show that the integral $\int_{-4}^{2} \frac{2}{x^6}\,dx$ does not exist.

Solution

As we'll see more formally in Calc 2, the area under the curve that the integral represents is infinite.

```
%-------------------------------------------------
% Stewart 398/35
%
syms x
int(2/xˆ6, x, -4, 2)

ans =

Inf


%
echo off; diary off
```

s398x43

Show that $\int_{\pi/2}^{\pi} \sec x \tan x \, dx$ does not exist.

Solution

Once again, we'll see in Calc 2 that the area under the curve that the integral represents is infinite.

```
%-------------------------------------------------
% Stewart 398/43: All at once
%
syms x
int(sec(x)*tan(x), x, pi/2, pi)

ans =

Inf


%
echo off; diary off
```